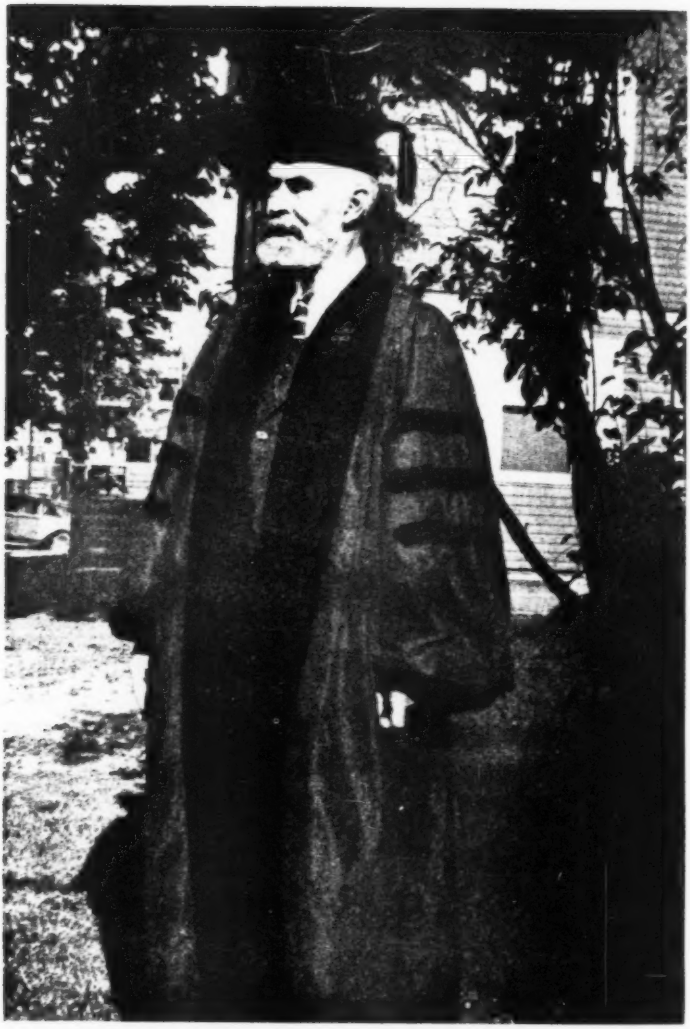

William Fogg Osgood
1864–1943

WILLIAM FOGG OSGOOD—IN MEMORIAM

With the death on July 22, 1943, of William Fogg Osgood, the American Mathematical Society has lost a beloved and distinguished member, and The Fraternity of Science a most faithful comrade.

William Fogg Osgood was born in Boston on March 10, 1864, the only child surviving beyond infancy of Dr. William Osgood, a physician and fourth in an unbroken line of that profession, and of Mary Rogers (Gannett) Osgood of Milton, Massachusetts. Both Osgoods and Gannetts were old New England families, the earliest American Osgood in the mathematician's line being John Osgood (1595–1651) who came to Ipswich, Massachusetts, in 1638 from Wherwell, near Andover (Hampshire), England, where the family's freehold seems to have antedated the Conquest. On the same side of the family, Osgood was a great great grandson of General Joseph Otis, the brother of James Otis the patriot.

Osgood attended the Boston Latin School, and entered Harvard in 1882, whence he graduated in 1886 summa cum laude with highest honors in mathematics and the second in a class numbering 286. In school his education had been centered in the classics, and at Harvard the study of the Greek authors formed a large part of his program; indeed he received second year honors in classics. But before long his interest in the experimental and mathematical sciences was aroused by his teachers at Harvard, most of all by his distant kinsman Benjamin Osgood Peirce; and his intention to devote his life to mathematics became established. After obtaining the A.M. in 1887, he received a travelling fellowship, and spent two years in Göttingen and a year at Erlangen where he obtained his Ph.D. (1890). He was awarded an L.L.D. by Clark University in 1909.

Osgood was married in 1890 to Therese Ruprecht (deceased) of Göttingen, and returned to become an instructor in the department of mathematics at Harvard (1890–93). He served there as assistant professor (1893–1903) and as professor (1903–33), and became professor emeritus in 1933. He was the father of three children: Dr. William Ruprecht Osgood of the Bureau of Standards, Mrs. Freda Osgood Silz (deceased), and Dr. Rudolph Osgood, a physician of Fall River, Massachusetts. In 1932 Osgood was married to Mrs. Céleste Phelps Morse. During 1934–36 he was visiting professor of mathematics at the National University of Peking. From 1936 until his death he lived in Belmont, Massachusetts.

Osgood was elected to the National Academy of Sciences in 1904. He was the eighth president of the American Mathematical Society

(1905–06). He was a member of the American Philosophical Society, the Deutsche Mathematiker-Vereinigung, the Leopoldinische-Carolinische Deutsche Akademie der Naturforscher, the Circolo Matematico de Palermo, corresponding member of the Mathematical Society of Charkow, the Göttinger Gesellschaft der Wissenschaften, and honorary member of the Calcutta Mathematical Society. He also belonged to Phi Beta Kappa and Sigma Xi.

We are fortunate in having an exhaustive bibliography of Osgood's works in the first volume of the Semicentennial Publications of this Society.[1] Moreover the original papers, numbering upward of four score, are too widely known and too technical to make more than their briefest mention desirable in this place. His study of term by term integration of non-uniformly convergent series (1896) contained in germ ideas of Borel measure. His example of a Jordan curve of positive area settled an important controversy. His work on conformal mapping is definitive. His treatment of the uniformization of algebraic functions formed an important contribution. Osgood initiated a branch of the calculus of variations. His work on the gyroscope showed how his mind could illuminate simple and important new aspects of a classical problem hitherto regarded as quite completed.

At the invitation of Felix Klein, Osgood wrote the article on the theory of functions of a complex variable in the Encyklopädie der Mathematischen Wissenschaften. Out of this grew his best known book, the *Lehrbuch der Funktionentheorie*. This book is a classic; it formed the first thorough and systematic treatment of the subject; it combines the deep impress of an individuality with the effective expression of a great mathematical tradition. The second volume deals with several complex variables and is today the only deep and broad systematization of the subject. It is a mine from which much material for future research will be drawn. So much of it is original that it has largely the import of a series of memoirs. During his last years, Osgood was working on the hypergeometric theory, which was intended to find a place in a later part of this book.[2]

In forming a picture of the mode and temper of Osgood's thought, three major influences should be regarded as having had a decisive effect in the formation of his mind: his predilection for physics, the early classical training, and the sojourn in Germany.

[1] R. C. Archibald, *A semicentennial history of the American Mathematical Society 1888–1938*, Amer. Math. Soc. Semicentennial Publications, vol. 1, New York, 1936, pp. 153–158.

[2] Circumstances have made it impossible as yet to ascertain how much has been left in a form suitable for publication.

Osgood once told me that he was at heart a physicist, and that if during his student days the career of physics had offered a more deeply mathematical and less disproportionately experimental form, he might well have made it his. And there was ever present in his mind the notion of mathematics finding one of its deepest justifications in serving as an instrument whereby the mind of man can penetrate into the mysteries of nature.

This is not to say that Osgood was any the less alive to that inner order which makes mathematics a fit subject of study in its own right. Not only did there appear to be nothing inconsistent between the two evaluations of the subject, but on the contrary he always saw the two—the extrinsic and the intrinsic justification of mathematics—as mutually reënforcing one another.

And this brings us to the second point. I have always felt that one of Osgood's most distinguishing characteristics was this sense of balance between values: applications and theory; traditional mathematics and innovations; teaching and research; and with it all the constant awareness that a mathematician is a human being who is using his human mind for his enlightenment, rather than a theorem-producing machine. And at the risk of being involved in a well known controversy, I hazard the opinion that this was due in no small degree to the fact that Osgood received his earliest intellectual impressions in the study of the classics. That sense of the dignity of reason, of tradition in excellence, and the instinct of historical continuity are qualities which are less to be derived from modern studies than from the writings of antiquity.

The third current of influence was the academic Germany of the eighteen-eighties, and especially the power of Felix Klein. The young Bostonian of four and twenty with his turn of mind at once scientific and classical came in Göttingen under the force of the great teacher whose ambition it was to unify mathematics and to set it forth in all its coördination and clarity. It is easy to realize the magnitude of the impression which was produced: the old scholar of the classics became the classical mathematician, and a flame was kindled which was to light a long career of science.

When Osgood and Bôcher returned from Germany to take positions on the Harvard faculty, they found the Department of Mathematics rather like that of a provincial college. It had contained individuals of eminence, but it could scarcely have supplied a real training in advanced modern mathematics. Then in a few decades it became one of the leading departments in the country, with a respectable place among the departments of mathematics in the world. This was

largely the natural result of the general movement of American education, and every member of the department at Harvard had his share in it. But if the credit of this rise to eminence can be given to any one man, Osgood was undoubtedly that one. His influence was exerted upon teaching at all levels, on the increased valuation of research, and perhaps most important of all, in making the many eminent appointments to the teaching staff—the only way in the long run in which any individual can assure the greatness of his department. Osgood was departmental chairman in 1918–22, and acting dean of the Graduate School in 1922 (February–July). He also belonged to various educational committees.

Osgood was one of the most notable teachers in American mathematics, and this at every grade from freshman to specialized graduate course. He was too true a democrat ever to allow himself to lower mathematics to the level of the student: it was his ambition on the contrary to raise the latter to the height of the subject. Ever speaking the language of the student, constantly mindful of his difficulties, Osgood spared no patient effort in bringing him to see what mathematics truly is: the powerful agent of illumination of the physical world, and the gem of human reason.

In his personal contacts with the students Osgood was a happy combination of the wise master and the cordial friend. This gentleman of the old school, whose charm and dignity made his acquaintance not only a pleasure but an honor, was so inclined to see the best in the men that he brought out their very best. Nor could it ever be ascertained whether a student's academic promise, or his being in trouble and in need of a friend constituted the greater claim to Osgood's kindness.

There are many with whom the memory of Osgood will dwell: With all those who have fallen heir to his intellectual heritage; and what state in the union fails to contain them? With his old students, to whom he brought the undistorted spirit of science; they must form by now a goodly host. With his contemporaries, whose ranks time has thinned, those companions in the task of bringing the Great Tradition to our shores. And also with some others, into whose lives he entered as a genial light and inspiration, beyond the power of language to convey.

BERNARD OSGOOD KOOPMAN

THE HIGHER COMMUTATOR SUBGROUPS OF A GROUP

REINHOLD BAER

It is not the object of this address to introduce you to new theories or to tell of great discoveries. Quite on the contrary; I intend to speak of unsolved problems and of conjectures. In order to describe these, certain concepts will have to be discussed; and for obtaining a proper perspective it will be necessary to mention a number of theorems, some of them new. The proofs of the latter will be relegated to appendices so that the hurried reader may skip them easily. The bibliography is in no sense supposed to be complete. We just selected convenient references for facts mentioned and beyond that just enough to be a basis for further reading.

1. **The hierarchy of invariant subgroups.** The subgroups of a group may be classified according to the operations which leave them invariant. There are first the *normal* subgroups of the group G, characterized by the fact that they are transformed into themselves by the inner automorphisms of G; and for this reason they had at one time appropriated the term "invariant subgroup." There are next the *characteristic* subgroups of G which are left invariant by every automorphism of G. Clearly not every subgroup of G is normal, unless G belongs to a comparatively special class of groups, the so-called abelian and hamiltonian groups; and neither is in general every normal subgroup characteristic, though this may happen too (for example, in cyclic groups and in simple groups).

These two classes of subgroups are well known, but for our purposes they are too big. There is next the class of subgroups which we shall term for lack of a better name *strictly characteristic*. A subgroup S of G belongs to this class if $S^f \leqq S$ whenever f is an endomorphism[1] of G and $G^f = G$. The distinction between characteristic and strictly characteristic subgroups does not cut very deep, since there exists a very big class of groups with the following property:

(Q) *If f is an endomorphism of G such that $G^f = G$, then f is an automorphism of G.*

This postulate (Q) is satisfied by every finite group and more generally by every group satisfying the ascending chain condition for normal subgroups. But the Q-groups are not exhausted by the groups

An address delivered before the Annual meeting of the Society in Chicago on November 27, 1943; received by the editors January 5, 1944.

[1] An endomorphism of the group G is a single-valued and multiplicative G to G function.

just mentioned, witness the free groups on a finite number of generators.[2] However, there exist groups that are not Q-groups, for instance the rational numbers modulo 1 and the free groups on an infinity of generators; and an example of a characteristic subgroup which is not strictly characteristic will be constructed in Appendix I.

Finally we have the *fully invariant* subgroups.[3] A subgroup S of the group G is termed fully invariant if $S^f \leqq S$ for every endomorphism f of G. Examples of groups whose centers are not fully invariant are easily constructed.[4] On the other hand it is readily verified that the center is always a strictly characteristic subgroup. For if z is an element in the center of the group G, and if $G^f = G$ is satisfied by the endomorphism f of G, then there exists to every element x in G an element y in G satisfying $x = y^f$; and we find

$$z^f x = z^f y^f = (zy)^f = (yz)^f = y^f z^f = xz^f$$

so that z^f is in the center too. Examples of fully invariant subgroups are obtained by forming the subgroup G^n generated by all the nth powers of elements in G, or the commutator subgroup (G, G) of G which is generated by all the commutators $(x, y) = x^{-1}y^{-1}xy$. More generally it may be said that the "word subgroups" in the sense of B. Neumann and P. Hall[5] are fully invariant whereas the "marginal subgroups" may only be shown to be characteristic.

The importance of characteristic subgroups stems from the fact that they themselves and their quotient groups are group invariants. But if we compare different groups, we have to have a method for deciding which are "corresponding" invariants of the different groups.[6] For it would not do to compare the structure of G^3 with the structure of H^5 or the order of G with the maximum order of the elements in H. We indicate one such possibility of comparing the fully invariant subgroups of different groups. If S is a subgroup of the group G, then we denote by $(G \to H; S)$ the subgroup of the group H which is generated by all the elements s^f for s an element in S and f a homomorphism of G into H. It is readily seen that $(G \to H; S)$

---

[2] For proofs of this theorem see F. Levi [1, p. 95] and W. Magnus [1, p. 276, VIII].

Criteria for Q-groups may be found in R. Baer [2], D. I. Fouxe-Rabinowitsch [1], and A. Malcew [1]. Numbers in brackets refer to the Bibliography at the end of the paper.

[3] This concept is due to F. Levi [1].

[4] See, for example, Appendix I below.

[5] P. Hall [2], B. Neumann [1].

[6] See in this context the functors discussed by Eilenberg and MacLane [2].

is always a fully invariant subgroup of H; and if S is a fully invariant subgroup of G, then we have

$$(H \to G; (G \to H; S)) \leqq S.$$

Equality we cannot always expect. But if G and H are both free groups on an infinity of generators, then equality holds, so that the fully characteristic subgroups of G and H form isomorphic partially ordered sets.[7]

Appendix I: Existence of characteristic, but not strictly characteristic subgroups

Let F be a finite group which is not abelian. Then F contains a cyclic subgroup T which is not part of the center of F. Denote by Z a cyclic group whose order equals the order of T and denote by $G(0)$ the direct product of F and Z. It is clear that Z is part of the center of $G(0)$ and that there exists an endomorphism $f(0)$ of $G(0)$ which maps F upon 1 and which effects an isomorphism of Z upon T.

Since $G(0)$ is a finite group, there exists a free group $G(1)$ on a finite number of generators and a homomorphism $f(1)$ of $G(1)$ upon the (full) group $G(0)$. Denote by $G(i)$ for $1<i$ groups isomorphic to $G(1)$ and by $f(i)$ an isomorphism of $G(i)$ upon $G(i-1)$ for $1<i$. Finally let G be the free product[8] of all the groups $G(i)$ for $0 \leqq i$. Then there exists[9] one and only one endomorphism f of G which induces $f(i)$ in $G(i)$ for $0 \leqq i$; and it is clear that $G=G'$.

Suppose now that B is a finite subgroup of G. Then it follows from a theorem on the subgroups of free products[10] that B is the free product of a free group V and of groups of the form[11] $B \cap x^{-1}G(i)x$. But B is a finite group and therefore it cannot contain a free group different from 1, nor can it be the free product of two groups different from 1. On the other hand free groups do not possess finite subgroups different from 1, since subgroups of free groups are free.[12] Thus we have shown the following fact.

If B is a finite subgroup of G, then $B \leqq x^{-1}G(0)x$ or $xBx^{-1} \leqq G(0)$ for suitable x in G.

Denote now by C the subgroup of G which is generated by the

[7] Baer [3].

[8] For the theory of free products see, for example, Baer and Levi [1].

[9] If we had formed the direct product of the groups $G(i)$ instead of forming their free product, then it would have been impossible to construct this endomorphism f.

[10] Baer and Levi [1, p. 392].

[11] By $S \cap T$ we designate the crosscut of the sets S and T.

[12] By Schreier's theorem.

center D of $G(0)$ and by all its conjugates in G. If g is an automorphism of G, then $G(0)^g$ is a finite subgroup of G. Hence it follows from the lemma just proved that $G(0)^g = y^{-1}G(0)y$ for some y in G. Consequently $D = yD^gy^{-1}$, since D is a characteristic subgroup of $G(0)$, and since the succession of g and of the inner automorphism $s \to ysy^{-1}$ of G induces an automorphism of $G(0)$. But from this fact one readily deduces that *C is a characteristic subgroup of G.*

From the theorems[13] on subgroups of free products one deduces now that C is the free product of D and of some conjugates to D in G. Thus D is a free factor of C and is consequently the crosscut of C and $G(0)$. This shows in particular that T is not part of C. But T is part of C', since $D' = D'^{(0)} = T$. Hence

C is not a strictly characteristic subgroup of G.

2. **Commutator subgroups.** If S and T are subsets of the group G, then we denote by (S, T) the subgroup generated by all the commutators $(s, t) = s^{-1}t^{-1}st$ for s in S and t in T. If S and T are fully invariant subgroups of G, then so is (S, T). Thus it is possible to use this operator for the inductive definition of fully invariant subgroups. We mention two important instances.

The derived series: $G^{(0)} = G$, $G^{(i+1)} = (G^{(i)}, G^{(i)})$.

The lower central series: ${}^0G = G$, ${}^{i+1}G = (G, {}^iG)$.

These are not the only possibilities. For instance, one could define a series $G(i)$ of fully invariant subgroups by the following recursion formulas:

$$G(0) = G, G(1) = (G, G), G(i+1) = (G(i), G(i-1)) \text{ for } 0 < i.$$

But so far only the derived and the lower central series seem to have proved interesting. A systematic method of constructing higher commutator subgroups has been evolved by P. Hall.[14] But his constructions are too involved to be sketched in a few words.

The calculus of commutator subgroups is based on the following simple formulas, which are easily verified by direct computation.[15]

$$(1) \qquad (x, y) = (y, x)^{-1}.$$

$$(2) \qquad (xy, z) = y^{-1}(x, z)y(y, z) = (x, z)((x, z), y)(y, z).$$

$$(3) \qquad (x, yz) = (x, z)z^{-1}(x, y)z = (x, z)(z, (y, x))(x, y).$$

If X, Y, Z are subgroups of the group G, then one deduces from these

[13] Baer and Levi [1, p. 396, Folgerung 3].

[14] Hall [1], in particular p. 43–48.

[15] Hall [1, p. 43].

formulas the following results.

$$(4) \qquad (X, Y) = (Y, X).$$

$$(5)^{16} \qquad (X, (Y, Z)) \leqq (Y, (Z, X))(Z, (X, Y)).$$

An induction argument using the definitions of derived and lower central series as well as formulas (4) and (5) leads now to the inequalities

$$(6)^{17} \qquad ({}^iG, {}^jG) \leqq {}^{i+j+1}G, G^{(i)} \leqq {}^{2^i-1}G.$$

Thus the derived series decreases much more rapidly than the lower central series. Another significant distinction between these two series is the fact that $G^{(i+1)}$ is a fully invariant subgroup of $G^{(i)}$ whereas ${}^{i+1}G$ need not be a fully invariant subgroup of ${}^iG$. A property common to both these series is the fact that the quotient groups of successive members are abelian.

Special properties of such series of fully invariant subgroups are invariant properties of the underlying groups and may therefore be used for characterizing special classes of groups. We give two examples of such properties.

(S) 1 *is the crosscut of the groups* $G^{(i)}$.

(N) 1 *is the crosscut of the groups* ${}^iG$.

It is an immediate consequence of formula (6) that every N-group is an S-group, though the converse is not true. In fact, finite S-groups are just the soluble groups whereas finite N-groups are exactly the nilpotent groups.[18] Finite nilpotent groups are known to be direct products of p-groups; and this theorem may be generalized as follows.

Every N-group without elements of order[19] 0 *is the direct product of (finite or infinite) p-groups.*[20]

In Appendix II we prove this theorem and we construct an example of a p-group P satisfying $P^{(2)} = 1$, though it is not an N-group, showing that the converse of the above theorem does not hold.

N-groups generated by a finite number of elements may be shown to be Q-groups (as defined in §1),[21] though there exist S-groups generated by two elements which are not Q-groups.[22]

---

[16] Hall [1, p. 47, Theorem 2.3].

[17] Hall [1, p. 53, Theorem 2.51 and p. 54, Theorem 2.54].

[18] See, for example, Baer [1], where further references may be found.

[19] An element in a group is said to be of order 0 if it generates an infinite cyclic group.

[20] A p-group is a group all of whose elements are of order a power of p.

[21] See, for example, Fouxe-Rabinowitsch [1].

[22] For proofs of these facts, cp. Baer [2].

Finally we indicate a method[23] for proving that a given group G is an N-group. Suppose that the ring R contains an identity element 1 (satisfying $r1=1r=r$ for every r in R), that 0 is the crosscut of the powers P^i of the two-sided ideal P in R, and that G is a multiplicative group of elements in R meeting the requirement:

$g \equiv 1$ *modulo* P *for* g *in* G.

If g is an element in G, then there exist uniquely determined elements g', g'' in P such that $g=1+g'$ and $g^{-1}=1+g''$; and g'' belongs to P^i if g' is an element in P^i. If g, h are elements in G, and if h', h'' belong to P^i, then

$$\begin{aligned}(g, h) &= (1+g'')(1+h'')(1+g')(1+h') \\ &\equiv 1+g''+h''+g'+h'+g''g'+h''h' \text{ modulo } P^{i+1} \\ &\equiv 1 \text{ modulo } P^{i+1}.\end{aligned}$$

Thus one is able to prove by complete induction the following fact: If g belongs to ${}^iG$, then $g \equiv 1$ modulo P^{i+1}. Consequently G itself is an N-group, since the crosscut of the ideals P^i is 0.

If T and $S \leqq T$ are normal subgroups of the group G, then we denote by $S \div T$ the set of all the elements x in G satisfying $(x, T) \leqq S$. It is readily seen[24] that the set $S \div T$ is a normal subgroup of G; and if S and T are characteristic subgroups of G, so is their *commutator quotient* $S \div T$. The best known example is the center $1 \div G$ of the group G or more generally:

The upper central series: $Z_0(G)=1$, $Z_i(G)=Z_{i-1}(G) \div G$ for $0<i$.

Interesting relations between upper and lower central series of a finite p-group have been discovered by P. Hall.[25] There are, however, fundamental differences, since the one series is descending, the other ascending, the one consists of fully invariant subgroups, the other of strictly characteristic subgroups which need not be fully characteristic.

In analogy to the definition of N- and S-groups one is led to the Z-groups, meeting the following requirement.[26]

(Z) *Every element in* G *is contained in at least one* $Z_i(G)$.

A finite group is known to be a Z-group if, and only if, it is an N-group. But in general no such relation holds. The non-abelian free groups are examples of N-groups with centers equal to 1, and there

[23] This method has been introduced by W. Magnus [1] who used it to prove that free groups are N-groups.

[24] Baer [3], Zassenhaus [1].

[25] Hall [1, p. 53, Theorem 2.51].

[26] Baer [1].

exist p-groups with this property.[27] An example of a Z-group, not an N-group, will be constructed in Appendix II.[28]

Appendix II: Nilpotency of N-groups

If x and y are elements of order a power of p in the N-group G without elements of order 0, then the order of xy is of the form ip^j where i is an integer prime to p. We denote by W the subgroup generated by x and y; and we put $(xy)^{p^j}=z$ so that z is an element of the order i prime to p. From ${}^kW \leqq {}^kG$ we infer that 1 is the crosscut of the subgroups ${}^kW$.

We are going to prove by complete induction with regard to k that $W/{}^kW$ is a finite group for every k. This is certainly true for $k=0$; and thus we assume its validity for $k-1$. Since W is generated by a finite number of elements, and since $W/{}^{k-1}W$ is finite, we deduce from the Reidemeister-Schreier method[29] that ${}^{k-1}W$ is generated by a finite number of elements. Consequently ${}^{k-1}W/{}^kW$ is an abelian group, generated by a finite number of elements whose orders are different from 0. This implies the finiteness of ${}^{k-1}W/{}^kW$, proving that $W/{}^kW$ is finite too.

Since $W/{}^kW$ is a finite group whose lower central series ends with 1, and since $W/{}^kW$ is generated by two elements of order a power of p, it follows from well known theorems[30] that $W/{}^kW$ is a p-group. Hence $z \equiv 1$ modulo ${}^kW$ for every positive k. Thus z belongs to the crosscut of the subgroups ${}^kW$. But this crosscut is 1, showing that $z=1$. Therefore we have proved the following fact.

If x and y are elements of order a power of p in the N-group G without elements of order 0, then xy is an element of order a power of p.

If we denote by $G(p)$ the set of all the elements of order a power of p in G, then it follows from the fact just established that $G(p)$ is a subgroup of G. Hence $G(p)$ is a fully invariant and therefore a normal subgroup of G. If p and q are different primes, then $G(p)$ and $G(q)$ are normal subgroups of G which have only the identity in common. Consequently $rs=sr$ for r in $G(p)$ and s in $G(q)$. Now it is readily seen that G is the direct product of its primary components $G(p)$, as we desired to prove.

The condition that the group G be an N-group, though indispensa-

[27] Baer [1, pp. 412, 413, Example 3.4].

[28] For another example of a Z-group, not an N-group, see Baer [1, p. 406, Example 2.5].

[29] See, for example, Baer [1, p. 396 (1.3)] or Zassenhaus [1, p. 108].

[30] See, for example, Zassenhaus [1, p. 107].

ble for the preceding proof, is not a necessary condition, as will be seen from the following *example of a p-group which is not an N-group.*

Denote by B the abelian group generated by elements b, $b(1)$, $b(2)$, $\cdots$, $b(i)$, $\cdots$, subject to the relations:

$$1 = b^p,\ b = b(i)^{p^i} \text{ for every positive } i.$$

It is readily seen that the crosscut B^{p^ω} of all the groups B^{p^i} is the cyclic group of order p, generated by b, and that B/B^{p^ω} is the direct product of a cyclic group of order p by a cyclic group of order $p^2 \cdots$ by a cyclic group of order $p^i \cdots$ and that the elements $b(i)$ represent a basis of B/B^{p^ω}.

Denote by $g(i)$ the automorphism of B which maps $b(i)$ upon $b(i)^{1+p}$ (and thus leaves b invariant) and which leaves all the $b(j)$ for $j \neq i$ invariant. It is obvious that the automorphism $g(i)$ is of order p^i and that the automorphisms $g(i)$, for positive i, generate an abelian group.

Let finally G be the group obtained by adjoining to B elements $t(i)$ subject to the relations:

$$1 = t(i)^{p^i},\ t(i)^{-1}xt(i) = x^{g(i)} \text{ for } x \text{ in } B \text{ and positive } i.$$

This group G is an extension of the group B by the group of automorphisms, generated by the $g(i)$'s.

Since G/B and B are both abelian p-groups, G itself is a p-group. Furthermore $G^{(2)} = 1$. One verifies inductively that ${}^iG = B^{p^i}$ and that the crosscut of the subgroups ${}^iG$ is therefore $B^{p^\omega} \neq 1$. Thus G has been shown to be a p-group, an S-group, but not an N-group.

Furthermore one verifies by complete induction that $Z_i(G)$, the ith term in the upper central series of G, is generated by the elements $t(j)$ for $0 < j < i$ and by the elements x in B which satisfy $x^{p^i} = 1$. Thus G is a Z-group.

3. **Burnside's problem.** Before the impact of topology on group theory made itself felt, group theory was concerned almost exclusively with the study of finite groups. Practically the first fact noticed by the student of finite groups is that the orders of the elements in a finite group are bounded and different from 0. This latter property defines a larger class of groups, which we are going to discuss now.

If the orders of the elements in the group G are bounded, then there exists the l.c.m. of the orders of all the elements in G. If m is this number, then $x^m = 1$ for every x in G. Every group G is the homomorphic image of a free group F. If N is the kernel[21] of the homomor-

[21] The kernel of the homomorphism f consists of those elements which are mapped upon 1 by f.

phism mapping F upon G, then $x^m=1$ for x in G is equivalent to[32] $F^m \leqq N$; and thus G is not only the homomorphic image of the free group F, but even of F/F^m. These *free groups reduced modulo m* may therefore be considered as universal groups for the class of groups under consideration. There is little known about their nature[33] and Burnside's problem might be stated somewhat loosely as the problem of determining the structure of F/F^m.

Let us start with a simple observation. The commutator (x, y) may be seen readily to equal

$$(x, y) = x^{-1}y^{-1}xy = (x^{-1}y^{-1}x)^2x^{-2}(xy)^2$$

so that we have in every group G the following relation:

$$(G, G) \leqq G^2.$$

This shows that F/F^2 is abelian and this makes it possible to answer completely any question we might have concerning the structure of these groups F/F^2.

It may easily be derived from a not at all trivial result of F. Levi and B. L. van der Waerden[34] that

$$^3G \leqq G^3;$$

and thus one is led to the question whether there exists to every positive integer n an integer n' such that

$$^{n'}G \leqq G^n.$$

The answer to this question is clearly in the negative, since we have noted before that $^{n'}G \leqq G^n$ implies that G/G^n is the direct product of its p-components whereas not every group satisfying $G^n=1$, for some n, is the direct product of its p-components. As a matter of fact a still stronger result may be proved.

If the free group F is not abelian, and if F/F^n is an N-group, then n is a power of a prime.

For a proof of this theorem see Appendix III. For prime powers n we prove in Appendix III a still stronger result; to wit:

If there exists an integer n' such that $^{n'}G \leqq G^n$ for every group G, then n is a prime.

---

[32] F^m is the subgroup generated by all the mth powers of elements in the group F.

[33] This is the reason why so much more emphasis is laid upon the commutator subgroups than upon these subgroups F^m.

[34] Levi and van der Waerden [1, p. 155].

Whether or not the converse of either of these theorems holds true is not known at present; and a decision would constitute an important step towards a solution of Burnside's problem.

The problem as Burnside himself conceived it was stated in a more limited fashion than we have considered it, since he imposed the additional restriction upon the groups under consideration that they should be generated by a finite number of elements. For these groups he has stated two famous conjectures.[35]

(I) *F/F^n is finite for every free group F of finite rank.*[36]

(II) *F/F^n is soluble for n odd and F a free group of finite rank.*

These conjectures are certainly true for $n=2$ and 3.

The conjecture (II) may be restated in a somewhat more precise form, since $(F/F^n)^{(i)}=(F^nF^{(i)})/F^n$.

(II′) *If n is an odd integer and if m is a positive integer, then there exists an integer $k=k(n, m)$ such that*

$$F^{(k)} \leqq F^n \text{ for } F \text{ a free group of rank } m.$$

A similar restatement of (I) is impossible because of the existence of simple groups. But for prime powers n this is possible, if one recalls that the lower central series of a finite p-group ends with the identity and that ${}^i(F/F^n)=({}^iF\,F^n)/F^n$; and thus we obtain the following conjecture.

(I′) *If n is a power of a prime, and if m is a positive integer, then there exists an integer $h=h(n, m)$ such that*

$${}^hF \leqq F^n \text{ for } F \text{ a free group of rank } m.$$

That (I′) and (II′) imply the finiteness of F/F^n is a consequence of the following easily verified fact. If the group G is generated by a finite number of elements, and if every element in G is of finite order, then $G/{}^iG$ and $G/G^{(i)}$ are both finite groups. But (I′) and (II′) for prime powers are essentially equivalent conjectures, as may be seen from the following theorem, a proof of which may be found in Appendix III.

If G is a p-group, generated by a finite number of elements, then there exist to every positive integer i integers i', i'' such that

$${}^{i'}G \leqq G^{(i)}, \quad G^{(i'')} \leqq {}^iG.$$

But in case these conjectures should not be verified, then this theo-

[35] Burnside [1].

[36] The rank $r(F)$ of the free group F is the number of elements in any free set of generators of F. It is at the same time the rank of the free abelian group $F/(F, F)$.

rem still asserts that *a p-group which is generated by a finite number of elements is an N-group if, and only if, it is an S-group.*

The importance of these statements may be seen from the following almost obvious equivalence.

If n is a power of a prime, and if F is a free group of finite rank, then the following two conditions are necessary and sufficient for finiteness of F/F^n.

(a) *F/F^n is an N-group.*

(b) *The lower central series of F/F^n is finite.*

But by the preceding remarks we may substitute for (a) and (b) the following conditions.

(a′) *F/F^n is an S-group.*

(b′) *The derived series of F/F^n is finite.*

If n is a prime power such that F/F^n is an N- (or S-)group for every free group F of finite rank, then one may verify the corresponding statement for every F/F^n with free F. Furthermore it is immediately seen that F/F^n is an N-group if, and only if, F^n is the crosscut of the subgroups $^iF\,F^n$ for all the integers i. The fact that free groups are N-groups may be considered an encouraging sign. But more is known. For Zassenhaus [2] proved a theorem from which it is easy to derive the fact that 1 is the crosscut of the groups $^iF\,F^{p^i}$ for p a prime and $i=1, 2, \cdots$. All this, however, does not suffice to prove (a). Thus it may be a consolation to know that it is possible to substitute for (a) and (a′) the following weaker conditions without impairing the validity of the above statements.

(a*) *If $(F/F^n)/{}^i(F/F^n)\neq 1$, then ${}^{i+1}(F/F^n)<{}^i(F/F^n)$.*

(a′*) *If $(F/F^n)/(F/F^n)^{(i)}\neq 1$, then $(F/F^n)^{(i+1)}<(F/F^n)^{(i)}$.*

Let us turn now to explaining the significance of condition (b) and its equivalent (b′). Denote by N the crosscut of all the subgroups ${}^i(F/F^n)$. For n a power of a prime, N is at the same time the crosscut of the subgroups $(F/F^n)^{(i)}$. The following fact is now readily verified.

If condition (b) *is satisfied by the prime power n, if the free group F is of finite rank, then the following conditions are necessary and sufficient for the group G to be a homomorphic image of the group $(F/F^n)/N$:*

(i) *G is finite.*

(ii) *$G^n=1$.*

(iii) *G may be generated by $r(F)$ elements.*

Condition (b) has been verified for $n=5$ and $r(F)=2$ by P. Hall.[37]

[37] For a proof see Grün [2].

Both conditions (a*) and (b) have been investigated recently with rather powerful tools.[38] So far, however, the results obtained have not been of that striking nature that lends itself to easy narration.

Appendix III: The lower central series of free groups modulo n

THEOREM 1. *If F is a non-abelian free group and n a positive integer such that F/F^n is an N-group, then n is a power of a prime.*

PROOF. Suppose that $n=n'n''$ is the product of the two relatively prime integers n' and n'' both of which are different from 1. Denote by S the direct product of n' cyclic groups each of which is of order n'', and let $s(1), \cdots, s(n')$ be a basis of S. There exists one and only one automorphism k of S which maps $s(i)$ for $0<i<n'$ upon $s(i+1)$ and $s(n')$ upon $s(1)$. This automorphism k is clearly of order n'. Consequently there exists a group T which is obtained by adjoining to S an element t subject to the relations:

$$t^{n'} = 1,\ t^{-1}st = s^k \text{ for } s \text{ in } S.$$

This group T is an extension of the abelian group S of order $n''^{n'}$ by the cyclic group T/S of order n'. If x is an element in T, then $x^{n'}$ is in S and $x^n=(x^{n'})^{n''}=1$, since $S^{n''}=1$. Furthermore it is readily seen that T may be generated by the two elements t and $s(1)$. But T is not the direct product of its primary components, since the two elements t and $s(1)$ of the relatively prime orders n' and n'' do not permute.

Since F is a non-abelian free group, its rank is at least 2 and consequently there exists a homomorphism h of F upon T. If H is the kernel of the homomorphism h, then we deduce $F^n \leqq H$ from $T^n=1$; and thus h induces a homomorphism of F/F^n upon T. Since T is not the direct product of its primary components, neither is F/F^n, as homomorphisms map primary components upon primary components. We proved in Appendix II that N-groups without elements of order 0 are the direct products of their primary components. Hence F/F^n is not an N-group if n is not a power of a prime.

THEOREM 2. *If there exists an integer n' such that $^{n'}G \leqq G^n$ for every group G, then n is a prime.*

PROOF. Suppose that n, n' are positive integers such that $^{n'}F \leqq F^n$ for every free group F of finite rank (this hypothesis which is weaker than the one enunciated in the theorem will be shown to be suffi-

[38] Grün [1], Magnus [1, 2], Witt [1], Zassenhaus [2].

cient). Then F/F^n satisfies $\mathfrak{n}'(F/F^n)=1$. Hence F/F^n is an N-group and it follows from Theorem 1 that n is a power of a prime.

It has been shown elsewhere[39] that there exists a group G with the following properties:

(i) Both $G/(G, G)$ and (G, G) are direct products of cyclic groups of order p.

(ii) The center of G is equal to 1.

From (i) one infers in particular that $G^{(2)}=1$ and that $G^{p^2}=1$. Thus there exists a representation of G as a quotient group of a free group: $G=F/N$ satisfying $F^{p^2}\leqq N$.

Suppose now that the prime power $n=p^m$ is not a prime, so that $1<m$. Then it follows from a preceding remark that $F^n\leqq N$ and that G is a homomorphic map of F/F^n. Suppose now that the element x in F belongs to $\mathfrak{n}'F$. Then there exists a free factor V of F which is a free group of finite rank such that x belongs to $\mathfrak{n}'V$. Now we may apply the hypothesis stated at the beginning of the proof, namely that $\mathfrak{n}'V\leqq V^n$. Thus x belongs to V^n and therefore to F^n, and we have shown: $\mathfrak{n}'F\leqq F^n\leqq N$. But this shows that the lower central series of $G=F/N$ is finite and ends with 1, a fact that is readily seen to contradict (ii). This contradiction proves that n is a prime.

Let us term the positive integer n *Burnsidean* if F/F^n is finite for every free group F of finite rank. Then we may prove the following converse of Theorem 1.

THEOREM 1′. *If n is a Burnsidean prime power, and if F is a free group, then F/F^n is an N-group.*

PROOF. If x is an element not 1 in F, then there exists a homomorphism h of F upon a free group V of finite rank such that $x^h\neq 1$, since F is the free product of two free groups V and W, the first one of which contains x and is of finite rank. This homomorphism h maps F^i upon V^i and $\mathfrak{i}F$ upon $\mathfrak{i}V$ for every positive i. If x were contained in the crosscut $\mathfrak{w}F$ of the subgroups $\mathfrak{i}F$, then x^h would belong to the crosscut $\mathfrak{w}V$ of the subgroups $\mathfrak{i}V$. But n is a Burnsidean prime power. Thus V/V^n is finite and of order a power of a prime. Hence there exists an integer w such that $\mathfrak{w}V\leqq V^n$. Since x^h belongs to $\mathfrak{w}V$, it belongs to $\mathfrak{w}V$ and therefore to V^n. But this implies that x belongs to F^n, and hence we have shown $\mathfrak{w}F\leqq F^n$, a fact that is equivalent to the assertion: F/F^n is an N-group.

We note that it would have been sufficient for our proof to make the hypothesis: F/F^n is an N-group for every free group F of finite rank,

[39] Baer [1, pp. 412, 413, Example 3.4].

a hypothesis that is considerably weaker than the assumption that n is Burnsidean.

As an application of Theorem 1′ we prove a converse of the theorem proved in Appendix II.

THEOREM 3. *Suppose that the positive integer n is the product of Burnsidean prime powers, and that the group G satisfies $G^n=1$. Then G is the direct product of its primary components if, and only if, G is the homomorphic image of an N-group H satisfying $H^n=1$.*

PROOF. It is a consequence of a theorem established in Appendix II that N-groups H satisfying $H^n=1$ are the direct products of their primary components. Homomorphisms map direct products of p-groups upon direct products of p-groups, proving the sufficiency of our condition.

Suppose conversely that G is the direct product of its primary components. If $n=p(1)^{m(1)}\cdots p(k)^{m(k)}$ with $0<m(i)$, $0<k$, then G is the direct product of groups $G(i)$ satisfying $G(i)^{n(i)}=1$ for $n(i)=p(i)^{m(i)}$. The group $G(i)$ may be represented as a quotient group $F(i)/M(i)$ of a free group $F(i)$; and clearly $F(i)^{n(i)}\leqq M(i)$. Thus $G(i)$ is the homomorphic image of $F(i)/F(i)^{n(i)}$. By hypothesis $n(i)$ is a Burnsidean prime power and thus it follows from Theorem 1′ that $R(i)=F(i)/F(i)^{n(i)}$ is an N-group. If R is the direct product of the N-groups $R(i)$, then $R^n=1$, R is an N-group and G is the homomorphic image of R, proving our contention.

THEOREM 4. *If G is a p-group, generated by a finite number of elements, then there exist to every positive integer i integers i', i'' such that*

$$^{i'}G\leqq G^{(i)},\quad G^{(i'')}\leqq {}^{i}G.$$

PROOF. It is a consequence of a theorem quoted in §1 that we may select as i'' any integer satisfying $i<2^{i''}$.

Since ${}^{i}G/{}^{i+1}G$ and $G^{(i)}/G^{(i+1)}$ are abelian groups the orders of whose elements are powers of the prime p, since such abelian groups are finite if they are generated by a finite number of elements, one proves inductively by the usual application of the Reidemeister-Schreier method[40] that $G/{}^{i}G$ and $G/G^{(i)}$ are finite groups. Thus $G/G^{(i)}$ is a finite p-group, and such groups are known to have the property that their lower central series is finite and ends with 1. Hence there exists an integer i' such that ${}^{i'}(G/G^{(i)})=1$ and this is equivalent to ${}^{i'}G\leqq G^{(i)}$.

---

[40] See footnote 29.

4. Invariants of the Hopf type. The group invariants discussed so far were subgroups and quotient groups of subgroups of the group under discussion. But there exist other possibilities of connecting groups with groups in an invariant fashion. An instance is the automorphism group of a group.

A group may be given either by inner properties or by some sort of representation. An example of the first kind of definition is the following one: the abelian group of order p^n the orders of whose elements are 1 and p. An example of the second kind is the customary definition of the symmetric group of degree n. Though it is certainly more desirable to characterize a group by inner properties, it is more common to define it by some sort of representation, and very often a suitable representation makes a group better accessible to treatment.

H. Hopf[41] has recently discovered a group invariant derived from a representation of a group as a quotient group. He showed that for normal subgroups N of free groups F the quotient group

$$(F,F)/(F,N)$$

depends only on the structure of F/N, and is an invariant of F/N. Now the groups F/F^n, discussed in §3, are defined by their representations only. Thus these Hopfian invariants and their generalizations may be a tool in their investigation.

We indicate a new proof of Hopf's invariance theorem which has the advantage of being a convenient basis for many important generalizations. To simplify notation we restrict ourselves to a discussion of the lower central series, which is quite typical.[42]

Suppose that M is a normal subgroup of the group H. Then H/M is a representation of a certain group G. We define inductively subgroups ${}_iM$ by the formulas: ${}_0M = M$, ${}_iM = (H, {}_{i-1}M)$. Then every ${}_iM$ is a normal subgroup of H, and we note the following inequalities:

$${}_iM \leqq {}^iH, \quad {}_iM \leqq {}_{i-1}M \text{ and } M \leqq {}_iM \div {}^iH.$$

1. Let us consider an endomorphism h of the group H, satisfying $x^h \equiv x$ modulo M. Of h we say then that it induces the identity in H/M. Assume now that we have shown already that h induces the identity in ${}^iH/{}_iM$. Then a simple application of formulas (2), (3) of §2 shows that h induces the identity in ${}^{i+1}H/{}_{i+1}M$.

2. Consider now two representations H/M and K/N of the same group G and homomorphisms h, k of H into K and K into H respectively with the following property: h induces an isomorphism a of

[41] Hopf [1].
[42] Baer [3].

H/M upon K/N and k induces the isomorphism a^{-1} of K/N upon H/M.

This hypothesis implies in particular $M^h \leqq N$ and $N^k \leqq M$; and one verifies readily that ${}^iH^h \leqq {}^iK$, ${}^iK^k \leqq {}^iH$, ${}_iM^h \leqq {}_iN$, ${}_iN^k \leqq {}_iM$. But hk induces the identity in H/M; and thus it follows from what we have shown under 1 that hk induces the identity in every ${}^iH/{}_iM$. From this fact and the previously stated inequalities one deduces immediately that h and k induce reciprocal isomorphisms between ${}^iH/{}_iM$ and ${}^iK/{}_iN$.

This implies in particular that the isomorphism a maps $({}_iM \div {}^iH)/M$ upon $({}_iN \div {}^iK)/N$.

3. We say now that two representations H/M and K/N of the group G are *similar* if every isomorphism between H/M and K/N may be induced by homomorphisms. Using this terminology the result obtained under 2 implies the following generalization of Hopf's invariance theorem: ${}^iH/{}_iM$ and $({}_iM \div {}^iH)/M$ are invariants of the class of similar representations of G to which H/M belongs.

It should be mentioned that not every representation belongs to a class of similar representations.

That this theorem is really a generalization of Hopf's theorem is a consequence of the fairly obvious fact that all representations of the group G as a quotient group of a free group (or of a free abelian group,[43] or of a free group reduced modulo n, and so on) are similar.

The range of applicability of these invariants seems to be fairly wide. So far they have been used mainly to show that homomorphisms meeting certain requirements are isomorphisms.[44]

As these invariants have not been defined in an invariant fashion, one may desire an invariant definition. This may be obtained either by using a normal form of the representation which is trivial as far as it can be done and not very interesting, or by a properly invariant definition which does not refer to any representation. This latter problem seems to be fairly deep and has been solved so far apparently in only one instance.[44]

Bibliography

R. Baer

1. *Nilpotent groups and their generalizations*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 393–434.
2. *Groups without proper isomorphic quotient groups*. To appear in Bull. Amer. Math. Soc.
3. *Representations of groups as quotient groups and their invariants*. In course of preparation.

[43] Eilenberg and MacLane [1, p. 765, Lemma 4.3].

[44] Eilenberg and MacLane [3].

R. Baer and F. Levi
1. *Freie Produkte und ihre Untergruppen*, Compositio Math. vol. 3 (1936) pp. 391–398.

W. Burnside
1. *On an unsettled question in the theory of discontinuous groups*, Quart. J. Math. Oxford Ser. vol. 33 (1902) pp. 230–238.

S. Eilenberg and S. MacLane
1. *Group extensions and homology*, Ann. of Math. (2) vol. 43 (1942) pp. 757–831.
2. *Natural isomorphisms in group theory*, Proc. Nat. Acad. Sci. U. S. A. vol. 28 (1942) pp. 537–543.
3. *Relations between homology and homotopy groups*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 155–158.

D. I. Fouxe-Rabinowitsch
1. *On the determinators of an operator of the free group*, Rec. Math. (Mat. Sbornik) N. S. vol. 7 (1940) pp. 197–206.

O. Grün
1. *Über eine Faktorgruppe freier Gruppen*. I, Deutsche Mathematik vol. 1 (1936) pp. 772–782.
2. *Zusammenhang zwischen Potenzbildung und Kommutatorbildung*, J. Reine Angew. Math. vol. 182 (1940) pp. 158–177.

P. Hall
1. *A contribution to the theory of groups of prime power order*, Proc. London Math. Soc. vol. 36 (1934) pp. 29–95.
2. *Verbal and marginal subgroups*, J. Reine Angew. Math. vol. 182 (1940) pp. 156–158.

H. Hopf
1. *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. vol. 14 (1942) pp. 257–309.

F. Levi
1. *Über die Untergruppen der freien Gruppen*, Math. Zeit. vol. 37 (1933) pp. 90–97.

F. Levi and B. L. van der Waerden
1. *Über eine besondere Klasse von Gruppen*, Abh. Math. Sem. Hamburgischen Univ. vol. 9 (1932) pp. 154–158.

W. Magnus
1. *Beziehungen zwischen Gruppen und Idealen in einem speziellen Ring*, Math. Ann. vol. 111 (1935) pp. 259–280.
2. *Über Beziehungen zwischen höheren Kommutatoren*, J. Reine Angew. Math. vol. 177 (1937) pp. 105–115.
3. *Neuere Ergebnisse über auflösbare Gruppen*, Jber. Deutschen Math. Verein. vol. 47 (1937) pp. 69–78.
4. *Über freie Faktorgruppen und freie Untergruppen gegebener Gruppen*, Monatshefte für Mathematik und Physik vol. 47 (1938) pp. 307–313.
5. *Allgemeine Gruppentheorie*, Enzyklopaedie der mathematische Wissenschaften I, 4, 9; 2. Aufl., 1939.

A. Malcew
1. *On isomorphic matrix representations of infinite groups*, Rec. Math. (Mat. Sbornik) N. S. vol. 8 (1940) pp. 421–422 (405–421).

B. H. Neumann
1. *Identical relations in groups*. I, Math. Ann. vol. 114 (1937) pp. 506–552.

K. Reidemeister
1. *Einführung in die kombinatorische Topologie*, Braunschweig, 1932.
E. Witt
1. *Treue Darstellung Liescher Ringe*, J. Reine Angew. Math. vol. 177 (1937) pp. 152–160.
H. Zassenhaus
1. *Lehrbuch der Gruppentheorie*, I, Hamburger Mathematische Einzelschriften vol. 21 (1937).
2. *Ein Verfahren, jeder endlichen p-Gruppe einen Lie-ring mit der Charakteristik p zuzuordnen*, Abh. Math. Sem. Hamburgischen Univ. vol. 13 (1940) pp. 200–207.

University of Illinois

THE NOVEMBER MEETING IN PASADENA

The four hundred first meeting of the American Mathematical Society was held at the California Institute of Technology on Saturday, November 27, 1943. The attendance was about one hundred thirty, including the following thirty-five members of the Society:

E. T. Bell, W. Z. Chien, P. H. Daus, R. P. Dilworth, H. J. Hamilton, F. C. Hinds, P. G. Hoel, D. G. Humm, D. H. Hyers, C. G. Jaeger, Glenn James, R. C. James, C. C. Lin, W. E. Mason, A. D. Michal, M. M. Mills, K. T. Millsaps, G. H. Peebles, Sam Perlis, W. C. Randels, William Shand, G. E. F. Sherwood, S. C. Snowdon, D. V. Steed, Alfred Tarski, A. E. Taylor, B. P. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, Morgan Ward, L. E. Wear, W. M. Whyburn, E. R. Worthington, Max Zorn.

After a brief opening session for contributed papers at which Professor A. E. Taylor presided, the meeting in the morning was devoted to the invited one hour addresses at which Professor T. Y. Thomas presided. By invitation of the Program Committee, Dr. Alfred Tarski of the University of California spoke on *Paradoxes of geometrical congruence* and Professor Max Zorn of the University of California at Los Angeles on *Mathematical induction.*

The Symposium on Applied Mathematics was held in the afternoon with Professor A. D. Michal presiding. Professor Theodore von Kármán of the California Institute of Technology and Northrop Aircraft Corporation spoke on *Generalized airplane wing theory*, Professor P. Y. Chou of National Tsing Hua University and the California Institute of Technology on *Velocity correlations and the solutions of the equations of turbulent fluctuation*, and Mr. Paul Lieber of the Douglas Aircraft Corporation on *Some applications of matrices to flutter and related phenomena.* In the absence of Mr. Lieber, the last address was read by Mr. K. Latzer of the Douglas Aircraft Corporation.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers 1 to 3 were read in the morning session while papers 4 to 6 were read by title. Paper 3 was read by Dr. Perlis.

1. R. P. Dilworth: *A decomposition theorem for partially ordered sets.* (Abstract 50-1-7.)

2. R. C. James: *Orthogonality and differentiability in normed linear spaces.* (Abstract 50-1-26.)

3. Roy Dubisch and Sam Perlis: *On the radical of a non-associative algebra.* Preliminary report. (Abstract 50-1-8.)

4. František Wolf: *On a problem of L. M. K. Boelter. Viscous fluid forced through a heated vertical pipe.* (Abstract 50-1-43-*t*.)

5. E. J. Purcell: *Variety congruences of order one in n-dimensional space.* (Abstract 50-1-47-*t*.)

6. K. T. Millsaps: *A note on generalized Hilbert space.* (Abstract 50-1-48-*t*.)

A. D. MICHAL,
Associate Secretary

BOOK REVIEWS

On growth and form. By D'Arcy W. Thompson. Cambridge University Press; New York, Macmillan, 1942. 1116 pp. $12.50.

It is a rare privilege to write a review of a new edition of a book which has already taken its place as one of the great classics of science. *On growth and form* by D'Arcy Thompson in this new edition is much as it was in the first. Its fundamental ideas have been very little revised, but a wealth of new material has been added, expanding it from 793 to 1116 pages.

To summarize its contents would be a very difficult matter. It is almost an encyclopedia of all the relations that have ever been discussed between mathematics and organic form. Among the subjects treated are: the form of the cell, tissues, concretions produced by living things, shells, horns, and teeth; from the dynamic point of view, growth and the relation between form and mechanical efficiency; and such perennial favorites of the geometrician as the form of the bee's cell and the arrangement of leaves.

For the most part the mathematics used in the book is elementary. Thompson makes no pretentions, but says he is using the tools he has, leaving it to better equipped workmen to carry on the work. Professor Archibald[1] in a review of the first edition deals more competently than I could with the mathematics. The general point of view seems more important than the mathematics itself. In fact the author frequently talks more about mathematics than in its language. He states his purpose in the introductory chapter (p. 14): "to correlate with mathematical statement and physical law certain of the simpler outward phenomena of organic growth and structure or form, while all the while regarding the organism, *ex hypothesi*, as a material and mechanical configuration." And in the Epilogue he concludes (p. 1096) "My task is finished if I have been able to show that a certain mathematical aspect of morphology, to which as yet the morphologist gives little heed, is interwoven with his problems, complementary to his descriptive task, and helpful, nay essential, to his proper study and comprehension of Growth and Form."

Various biologists have placed very different estimates on the value of the book. Professor Sinnott,[2] in his review, has been exceedingly generous and attributes to it a wide and important influence. Professor McClung[3] on the other hand has stated that its influence is slight. In

[1] R. C. Archibald, Bull. Amer. Math. Soc. vol. 24 (1918) p. 403.
[2] E. W. Sinnott, Quarterly Review of Biology vol. 18 (1943) p. 64.
[3] C. E. McClung, Science N. S. vol. 96 (1942) p. 471.

spite of the fact that the book has been very much admired, I am afraid that Professor McClung's estimate is nearer the truth. The ideas of *On growth and form* have played little part in the spectacular advances of biology since the book was first written. For this I think there is a very good reason: the point of view which the book represents is out of fashion, and is indeed the antithesis of the one now in vogue, to which these advances have been due.

When the book was first published in 1917, experimentation in genetics was just beginning to produce its brilliant results and experimental embryology was approaching its most spectacular era. Here were exciting things to attract the young investigator with which the abstract ideas of the mathematician, whose language he all too frequently could not understand, could hardly compete. A rich harvest has been reaped in these fields with very slight if any influence of Thompson's book. Interestingly enough, on the other hand, almost every reviewer has complained that the new edition has not been influenced by this research, which is for the most part hardly mentioned. McClung[3] misses a treatment of cytogenetics, Buchanan[4] a treatment of physiological gradient ideas, and both Sinnott[2] and Hammett[5] list a number of important subjects that one might expect to have been treated.

The failure to take the results of modern biological research into consideration (Sinnott says the revision might just as well have been written twenty-five years ago) is related to the failure to make an important impact upon this research. They are both due to the antithesis of the fundamental ideas. *On growth and form* harks back to an older habit of thought, and may also as Wrinch[6] suggests be the herald of a new era, but it is not an essential part of contemporary biological advance.

There are only three essentially different ways of considering biological form. It may be considered as present from the beginning, existing either as an actual minute replica in the germ, or as an "idea" of some nonphysical formative entity like the entelechy of Aristotle and Driesch; or existing neither actually nor ideally, but merely potentially, in the organization of the germ. In older biological thinking form always pre-existed in some fashion or other. The adult form was in existence from the beginning of the life of the organism as an individual. The "preformationists" of the 18th century—their ideas culminating in Bonnet—believed that the germs of all organisms ever

[4] J. W. Buchanan, Physiological Zoology vol. 16 (1943) p. 135.
[5] F. S. Hammett, Growth vol. 7 (1943) p. 321.
[6] D. Wrinch, Isis vol. 25 (1943) p. 232.

to exist were created at the original creation. Each germ was a minute replica of the adult that was to develop from it. It had merely to expand. There was no real development of form. It had existed from the beginning.

The "epigeneticists," their opponents, believed that organisms are developed from unorganized and unformed matter under the influence of a vital, nonphysical entity. Here the form of the adult preexisted not actually, as in the germ of the preformationist, but as an idea of the entelechy or *vis essentialis*. The unformed material was moulded into shape by this agent—the actual form produced was new, an epigenesis.

With the development of modern techniques it became clear that the epigeneticists were right in that the germ contains no actual replica of the adult. The form develops rather by gradual stages. The notion of a nonphysical formative agent has not appealed to most biologists, however, although Driesch has made a valiant attempt to sustain it. The alternative chosen by most is a modified form of preformationism, in which the germ is considered to contain not a preformed replica of the adult, but an organization which determines the development of the adult form. The nature of this organization may be considered from two points of view: biologically, from the point of view of heredity, as with the trend begun by Weismann and leading to modern genetics, or mathematically and physically as in Thompson's book.

Thompson says (p. 1022) "To look on the hereditary or evolutionary factor as the *guiding principle* in morphology is to give to that science a one-sided and fallacious simplicity" and states his position unequivocally (p. 340): "The efforts to explain 'heredity' by the help of 'genes' and chromosomes, which have grown up in the hands of Morgan and others since this book was first written, stand by themselves in a category which is all their own and constitute a science which is justified of itself. To weigh or criticize these explanations would lie outside my purpose, even were I fitted to attempt the task. . . . I leave this great subject on one side not because I doubt for a moment the facts nor dispute the hypotheses nor decry the importance of one or other; but because we are so much in the dark as to the mysterious field of force in which the chromosomes lie, far from the visible horizon of physical science, that the matter lies (for the present) beyond the range of problems which this book professes to discuss, and the trend of reasoning which it endeavors to maintain."

Thompson's concept of organic form is that it is predetermined by

the physical organization of the system in which it develops, and "it is in obedience to the laws of physics that their particles have been moved, moulded and conformed" (p. 10). The nature of the physical organization and of the physical processes by which the form is produced from it is to be arrived at by mathematical analysis and physical interpretation of the adult form (p. 16). "The form of an object is a 'diagram of forces,' in this sense, at least, that from it we can judge of or deduce the forces that are acting or have acted upon it." "In an organism, great or small, it is not merely the nature of the motions of the living substance which we must interpret in terms of force (according to kinetics), but also the conformation of the organism itself, whose permanence or equilibrium is explained by the interaction or balance of forces, as described in statics." His interest is not in the biological analysis of the organization of the germ, from the point of view of either the geneticist or the experimental morphologist. He approaches the problem from the opposite direction.

He thinks of form as a Platonist. In his discussion of the tortoise shell, for example (p. 517), he is obviously talking of an ideal tortoise, more valid than any actual specimen or species, and any variant from which may be considered an "accident." As a matter of fact in the Epilogue (p. 1097) he alines himself with the teaching of Plato and Pythagoras, and again (p. 1094) he says "In natural history Cuvier's 'types' " (which are Plato's ideas under another name) "may not be perfectly chosen nor numerous enough, but types they are; and to seek for stepping stones across the gaps is to seek in vain, forever." On this view his attitude toward embryology (p. 5) and his failure, pointed out by Julian Huxley,[1] to apply his method of transformation of coordinates to stages in the life history of a single organism, are readily explainable. He is not thinking of the development of form as a biological process at all. He takes the adult form as given and analyzes it.

To place Thompson thus among the Platonists and beside the transcendental morphologists of the last century is by no means derogatory. Man has been seeking systematically for answers to his great problems for only 5000 years at most, and on a conservative estimate will be doing so for some 5,000,000 years to come. To attempt to read off with finality any consistently developed line of thought would seem at this stage presumptuous. It needs to be strongly emphasized that none of the hypothetical entities proposed on the basis of experiments in embryology deal with form in the sense that Thompson does. The gradients of Child and the fields of Weiss

[1] J. S. Huxley, *Problems of relative growth*, New York, 1932, p. 105.

serve to organize many significant facts about development, but do not tell us why specific form develops. Child[8] says that the specific form develops because the gradient operates in a substratum of specific protoplasm, but what does that mean? Weiss[9] says "the fact that each cell is bound to react exclusively in accordance with the standards of the species to which it belongs" constitutes "the principle of 'genetic limitation.'" What is the basis of that? In spite of all we have learned about the nature and action of organizers and evocators, Needham admits "that after all the larger part of the mystery remains in that we can as yet form little idea of what constitutes reactivity—competence to react to the morphogenetic inductor."[10] The specificity of protoplasm from the point of view of morphogenesis, and the competence of tissues to respond in a specific way to organizers are associated with the "form" in Thompson's *On growth and form*. The present day experimental embryologist is simply not investigating it at all. The geneticist, with his genom—the fundamental organization of the germ made up of genes arranged on chromosomes—comes nearer to it. It may well be, as Wrinch[6] suggests, that as the chemist learns more of the morphology of huge molecules and their aggregates, and as the biologist learns to apply this knowledge to the structure of the genom, a new place will be found, in our thinking, for the form of organisms as the morphologist, who descends from the line of Plato, thinks of it.

Even before that time arrives, however, this book has a message for the biologist, whose idea of quantitative biology is frequently merely the statistical treatment of data, sometimes even forgetting that "measurements may be as empty of significance as any other kind of descriptive materials. The statistical answers that can be wrung out of such measurements may have very little meaning if the quantities measured depend on a multiplicity of causes. Statistical treatment may indicate that certain results are significant, but what they signify no man may know, or even surmise."[11] Such a man might gain from *On growth and form* an idea of what could be meant by mathematical biology.

For the mathematician who has an interest in the relationship between his abstract ideas and the phenomena of the natural world, the book should be a treasure house. The author attributes to Lobatchevsky the statement (p. 11) that "there is no branch of mathematics

[8] C. M. Child, *Individuality in organisms*, Chicago, 1915, p. 188.
[9] P. Weiss, *Principles of development*, New York, 1939, p. 362.
[10] J. Needham, *Biochemistry and morphogenesis*, Cambridge, 1942, p. xiii.
[11] T. H. Morgan, *Experimental embryology*, New York, 1927, p. 13.

however abstract, which may not some day be applied to phenomena of the real world."

The book is written in an elegant, almost poetic, style that makes it delightful to read, and its many references conveniently placed at the bottom of the page near the related subject matter serve as a lure to further reading.

J. WALTER WILSON

Partial differential equations. By Frederick H. Miller. New York, Wiley; London, Chapman and Hall, 1941. 9+259 pp. $3.00.

According to the author the book is intended to be a text in a first course in partial differential equations. The chapter on ordinary differential equations is intended for review and reference purposes and not as a first course in the subject. The author finds it advisable to include a chapter on direction cosines and partial derivatives, probably because so many exercises in the book are taken from geometry. In the main the book is concerned with the quest for solutions depending on arbitrary constants and arbitrary functions. The examples of Chapter III show very clearly why this viewpoint of the subject is much more complicated in the case of partial differential equations than it is in the case of ordinary equations. In ordinary equations the solution of an nth order equation depends on n arbitrary constants, and conversely, the elimination of n arbitrary constants leads to an equation of nth order. In general, the number of partial derivatives of a given order is higher than the number of independent variables. The elimination of two arbitrary functions may lead to a pair of third order equations in one unknown. Since the first order partial differential equation behaves more like an ordinary equation than do those of higher order, Chapter IV on the linear equation of first order and Chapter V on nonlinear equations of first order are almost entirely devoted to the quest for solutions depending on arbitrary functions and arbitrary constants.

Chapter VI on Fourier series and the boundary value problems in Chapter VII furnish an exception to the above viewpoint. In this work the author is, of course, not seeking solutions depending on arbitrary functions. Chapter VI on Fourier series contains a statement of the expansion theorem for a function continuous except for a finite number of jump discontinuities. Chapter VII on the linear equation of higher order is devoted largely to the consideration of operator methods, undetermined coefficients, and variation of parameter methods for obtaining the particular solution. In case the nth order differential operator can be factored into linear factors a complimentary

function depending on n arbitrary functions can be obtained. As mentioned above, Chapter VII also contains separating variables and expansion of boundary conditions in Fourier series. The equations treated include the vibrating string, the cable, fluid flow, and heat flow. These equations have already been derived from physics in Chapter III.

Chapter VIII on nonlinear equations of second order is devoted largely to obtaining solutions of Monge's equation depending on arbitrary functions.

The book contains very few examples from physics except in the boundary value problems solved with Fourier series. However, the book contains an exceptionally large number of problems in which the student is asked to find a solution depending on arbitrary constants or functions. In a large proportion of these the partial differential equation is proposed as a problem in geometry.

The multiple integration approach to partial differential equations is not touched upon. The concept of characteristics of a second order equation and the classification of second order equations does not appear. Also there is no mention of successive integrations.

EDWIN W. TITT

The calculi of lambda-conversion. By Alonzo Church. (Annals of Mathematics Studies, no. 6.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1941. 2+77 pp. $1.25.

This is a brief and attractively written introduction to the remarkable formal systems discovered by Church and called by him calculi of lambda-conversion. These systems were developed by Church in collaboration with his students, S. C. Kleene and J. B. Rosser. The present booklet, which is lithoprinted, is in most respects a considerable improvement over the same author's mimeographed Princeton lecture notes of 1936, of which it may be considered a revision. The notation has been simplified and improved, and the treatment of Gödel numbers is much simpler than in the former version. The proof of the fundamental Church-Rosser consistency theorem is now given in full detail, and the section on recursive arithmetic has been considerably expanded.

Nevertheless the text of the present version totals only 71 pages. This brevity is partly accounted for by the plan which the author has wisely adopted of considering only the calculus of lambda-conversion proper in full detail. In fact, the first four chapters are devoted to this, the most elementary of the lambda-calculi. The more com-

plicated, but also more useful, systems called the calculus of λ-*K*-conversion, the calculus of restricted λ-*K*-conversion, and the calculus of λ-δ-conversion, as well as the applications of the latter to symbolic logic, are discussed only very briefly in the fifth and final chapter.

The first chapter is introductory, and explains the intended interpretation of the symbols by means of examples. The second chapter presents the lambda-calculus as a formal system; and a very simple formal system it is. There are three kinds of symbols (the symbol λ, parentheses, and letters for variables), one undefined operation called *application* and denoted by juxtaposition, and three rules of procedure called the rules of lambda-conversion. There are no axioms and no theorems, the subject being developed metatheoretically. (This is a matter of convenience; it would be possible to introduce axioms in terms of a symbol for interconvertibility of formulas, if that were considered desirable.) Chapter II also deals with the important notion of *normal form*, and contains the proof of the Church-Rosser consistency theorem. The author intends only those formulas which have a normal form to be meaningful, although there exist quite simple formulas with no normal form and hence with no interpretation. It has been shown that no effective procedure exists for determining in all cases whether a given formula has a normal form. This fact is related to the incompleteness property which, according to the results of Gödel, any system which is consistent and adequate for formal logic must have.

In the third chapter the positive integers are defined in terms of the lambda symbolism. This is possible because the system contains formulas which, applied to a function, have the effect of iterating it a given number of times. The familiar operations of addition, multiplication, and exponentiation of integers are also defined. It is shown that an integral-valued function of positive integers is general recursive if and only if it is lambda-definable, that is, definable in terms of the lambda symbolism. Reasons are given for identifying the notion of an effectively calculable function of positive integers with that of a lambda-definable function, or equivalently of a general recursive function.

Chapter four deals with Gödel numbers. It is convenient, in formal systems which contain a notation for the integers, to develop a method of numbering the formulas of the system which is definable in terms of the system itself. This enables the system to talk about itself, roughly speaking, and sometimes, indeed, even to contradict itself (but that cannot happen here). Since it is simpler to assign the same Gödel number to two formulas if they differ only in the letters

used to represent bound variables, the author first assigns to every well-formed formula a special kind of formula called a *combination*, and then enumerates the combinations. Formulas which differ only in the symbols for bound variables are assigned the same combination. In connection with combinations, the close connection between the lambda-calculi and the combinatory logic of M. Schönfinkel and H. B. Curry is brought out.

The restricted calculus of lambda-conversion which is treated in the first four chapters is not an adequate basis for logic or mathematics. In particular, constant functions cannot be defined because of the rule that $\lambda x M$ (which is interpreted to mean the function that M is of x) is not well-formed unless x occurs as a free variable in M. The fifth chapter deals briefly with modified systems which avoid the objection mentioned (the lack of constant functions), but which give rise to new difficulties, as the author points out. Of these, the calculi of λ-K-conversion and of restricted λ-K-conversion arise by changing the definition of well-formed formula to allow more well-formed formulas. The λ-δ-calculus involves the introduction of a new symbol δ which behaves somewhat like a sign of equality.

Finally the author outlines his method of defining within the λ-δ-calculus a system of symbolic logic. This is accomplished by identifying the truth values *truth* and *falsity* with the formulas for the integers 2 and 1 respectively. The provable formulas of the logic are then those formulas of the calculus which are convertible to the formula for the integer 2, that is to the formula for truth, according to the rules of λ-δ-conversion. Operations of negation, conjunction, and disjunction are defined, as well as an existential quantifier and a selection operator. All these have properties differing from those of the corresponding operations of classical logic, and resembling somewhat those of the intuitionist logic. It is indicated that there are difficulties involved in defining a universal quantifier. Although this logic seems peculiar in some ways, it has the advantage over certain other systems that there exists for it an elementary consistency proof, namely the analogue for the λ-δ-calculus of the Church-Rosser theorem. The further elaboration of this system of logic is left for another book.

In this book Church has done a first-rate job of exposition, which should be examined by everyone interested in the foundations of mathematics. Not that it is all easy reading; that would be impossible in the nature of the subject. The fundamental notions of the lambda-calculus have what seems to be a simple interpretation, but this interpretation does not seem to help one to follow the details. Perhaps this is partly because of the long arrays of symbols required to

define most of the important formulas. Verification that the formulas have the desired properties must proceed formally according to the conversion rules, and the work is often tedious.

Two interesting points concerning these calculi may be noted. One is that arithmetic is developed first, and logical notions are then defined in terms of arithmetic, whereas in most such formal systems, like the Whitehead-Russell system, logical notions come first and arithmetic is defined in terms of logic. The other point is that these calculi seem to support the views of the effectivists (including the intuitionists) who would exclude from mathematics the axiom of choice and in general those concepts which are not constructively definable. In particular the prominent role played here by the integers is suggestive of the viewpoint of Kronecker. However, it should be said that although the lambda-calculi make it possible to state more exactly what is meant by an effective definition, and to arrive at interesting results concerning the concept of effectiveness, one can use the methods of this book without committing oneself to the effectivist position. Presumably it is quite possible to erect a non-effectivist mathematics on the basis of these systems.

Church has suggested elsewhere that his lambda notation might be useful in avoiding the ambiguity which results (especially in dealing with function spaces) from the bad habit of using the notation $f(x)$ to stand sometimes for a function as a single entity, and sometimes for the value of the function when the argument is x. It is interesting to note that Whitehead and Russell use the notation $f(\hat{x})$ for the first of these concepts in place of Church's $\lambda x \cdot f(x)$ or $\lambda x(fx)$, and that our elementary textbooks at least provide for the distinction by the use of different letters for the argument; thus $f(x)$ stands for a function, while $f(a)$ or $f(x_0)$ stands for a functional value. Unfortunately the distinction is not always made. It should of course be said that Church's lambda notation is greatly superior in some ways to these other notations, and allows the construction of formulas not representable in them.

ORRIN FRINK, JR.

NOTES

The supply of certain issues of early volumes of the BULLETIN is almost exhausted. Any member who has copies of the particular issues listed below which he would be willing to contribute or sell is invited to communicate with the office of the Society at 531 West 116th Street, New York City, 27. Bulletin of the New York Mathematical Society, vol. 2, no. 3; Bulletin of the American Mathematical Society, vol. 1, nos. 3, 5; vol. 2, nos. 2, 8; vol. 3, no. 3; vol. 4, nos. 8, 9, 10; vol. 5, no. 6; vol. 8, nos. 6, 8, 9, 10; vol. 9, nos. 1, 3; vol. 10, nos. 1, 3, 4.

The BULLETIN has received from Professor A. Terracini and Professor F. Cernuschi, editors of Matematicas y Fisica Teorica, published in Tucuman, Argentina, an invitation for mathematical contributions. Accepted papers will be published in the language in which they are written.

The Sylvester Medal of the Royal Society has been awarded to Professor J. E. Littlewood of the University of Cambridge.

Dr. Vannevar Bush, President of the Carnegie Institution of Washington, has been awarded the Edison Medal for 1943 by the American Institute of Electrical Engineers.

Professor J. L. Synge of Ohio State University has been awarded the Henry Marshall Tory Medal of the Royal Society of Canada.

President H. N. Davis of the Stevens Institute of Technology has been elected an honorary member of the Institution of Mechanical Engineers of Great Britain.

Professor E. T. Whittaker has been elected president of the Royal Society of Edinburgh.

Professor Helen M. Walker of Teachers College, Columbia University, has been elected president of the American Statistical Association.

Professor J. L. Walsh of Harvard University has been elected vice president of the American Association for the Advancement of Science.

Assistant Professor A. F. Bernhart of Bucknell University has been appointed to an assistant professorship at the University of Oklahoma.

Professor G. D. Birkhoff of Harvard University is on leave of absence for the present winter term in order to give a series of lectures at the University of Mexico.

Assistant Professor M. G. Boyce of Western Reserve University has been promoted to an associate professorship.

Assistant Professor A. T. Brauer of the University of North Carolina has been promoted to an associate professorship.

Assistant Professor H. A. Dangel of the University of Cincinnati has been promoted to an associate professorship.

Miss Flora Dinkines of the University of South Carolina has been promoted to an adjunct professorship.

Mr. Edwin Eagle of Taft Junior College, Taft, California, has been appointed to an assistant professorship at Oregon State College.

Associate Professor Paul Eberhart of Washburn University, Topeka, Kansas, has been promoted to a professorship.

Mr. W. E. Ekman of the University of South Dakota has been promoted to a professorship.

Dr. W. S. Erickson of St. Olaf College, Northfield, Minnesota, has been appointed to a professorship at Minot State Teachers College, Minot, North Dakota.

Mr. M. P. Fobes of the College of Wooster, Wooster, Ohio, has been promoted to an assistant professorship.

Mr. W. A. Gager of the Junior College in St. Petersburg, Florida, has been appointed to an associate professorship at the University of Florida.

Dr. M. C. Hartley has been appointed to an assistant professorship at the University of Illinois.

Associate Professor H. H. Hartzler of Goshen College, Goshen, Indiana, has been promoted to a professorship.

Assistant Professor R. N. Haskell of the University of Texas has been promoted to an associate professorship.

Assistant Professor P. R. Hill of the University of Georgia has been promoted to an associate professorship.

Associate Professor J. W. Hurst of Montana State College has been promoted to a professorship.

Associate Professor J. A. Hyden of Vanderbilt University has been promoted to a professorship.

Assistant Professor F. B. Jones of the University of Texas has been promoted to an associate professorship.

Assistant Professor A. H. Knebel of the University of Cincinnati has been promoted to an associate professorship.

Dr. Fulton Koehler of the Northwest National Life Insurance Company has been appointed to an assistant professorship at the University of Minnesota.

Mr. J. E. LaFon of the University of Oklahoma has been promoted to an assistant professorship.

Assistant Professor R. G. Lubben of the University of Texas has been promoted to an associate professorship.

Dr. C. A. Ludeke of the University of Cincinnati has been promoted to an assistant professorship.

Mr. J. N. McClelland of Drake University has been promoted to an assistant professorship.

Dr. J. C. C. McKinsey of New York University has been appointed to an assistant professorship at Montana State College.

Assistant Professor W. T. MacCreadie of Bucknell University has been promoted to an associate professorship.

Professor A. E. Meder, Jr., of the New Jersey College for Women has been appointed secretary of Rutgers University.

Associate Professor C. E. Melville of Clark University has been promoted to a professorship.

Professor H. A. Meyer of Hanover College, Hanover, Indiana, has been appointed acting professor at Indiana University.

Assistant Professor R. R. Middlemiss of Washington University has been promoted to an associate professorship.

Mr. E. T. Miller of the University of Cincinnati has been promoted to an assistant professorship.

Associate Professor H. L. Miller of the University of Cincinnati has been promoted to a professorship.

Professor F. D. Murnaghan of Johns Hopkins University has been appointed to a visiting professorship at Brown University for the current year.

Associate Professor W. G. Pollard of the University of Tennessee has been promoted to a professorship.

Mr. W. E. Restemeyer of the University of Cincinnati has been promoted to an assistant professorship.

Mr. H. L. Rice of the University of Omaha has been promoted to an assistant professorship.

Assistant Professor E. B. Roessler of the College of Agriculture of the University of California has been promoted to an associate professorship.

Associate Professor J. B. Rosser of Cornell University has been promoted to a professorship.

Miss S. Grace Smyth of Knox College has been promoted to an associate professorship.

Mr. A. L. Starrett of Georgia School of Technology has been promoted to an assistant professorship.

Associate Professor W. R. Talbot of Lincoln University, Jefferson City, Missouri, has been promoted to a professorship.

Dr. H. C. Trimble of Iowa State Teachers College has been promoted to an assistant professorship.

Assistant Professor P. L. Trump of the University of Wisconsin has been promoted to an associate professorship in the teaching of mathematics.

Mr. R. N. Van Arnam of Lehigh University has been promoted to an assistant professorship.

Dr. G. L. Walker of the University of Delaware has been promoted to an assistant professorship.

Miss Fern Welker of the University of Toledo has been promoted to an assistant professorship.

Dr. Y. K. Wong of the University of Chicago has been appointed lecturer at the University of North Carolina.

The following individuals are assisting in the military training program at the University of Chicago: Professors J. M. Kinney and E. R. Breslich, Miss Evelyn R. Garbe, Messrs. Herman Meyer, Daniel Zelinsky, H. J. Zimmerberg, L. A. Blatz, Charles Brumfiel, Morris Friedman, Miss Marjorie Groves, Messrs. A. R. Jacoby and Charles Nichols.

The following appointments to instructorships are announced: University of Cincinnati: Mr. Robert Fopma; DePaul University: Mr. Paul D'Arco; Greenbrier Junior College, Lewisburg, W. Va.: Miss Margaret R. Davis; Heidelberg College, Tiffin, Ohio: Dr. Ruth G. Simond; Kansas State College: Mr. D. K. Brooks, Mr. L. E. Milleson, Mr. W. V. Unruh; University of Kentucky: Mrs. J. C. Lamb, Mrs. C. G. Latimer; Lehigh University: Mr. R. C. King, Mr. S. W. Smith; Michigan State College: Mr. Nicholas Musselman; Ohio State University: Mr. L. H. Miller; University of Oklahoma: Miss Phyllis Barclay, Mrs. Dorothea A. Sudduth; Oregon State College: Miss Florence Bakkum, Mr. L. R. Foote, Mr. Fred Young; Pennsylvania State College: Mr. J. R. Kinney, Mrs. Nellie M. Krall; University of Pennsylvania: Mr. R. C. Cambell; Purdue University: Mr. H. M. Anderson, Mr. F. C. Leone; Rice Institute: Mr. G. R. MacLane; Stanford University: Miss Marjorie L. Hoffman, Miss Mary V. Sunseri; Yale University: Mr. J. S. Blair, Mr. J. C. Caughlan, Dr. W. C. G. Fraser, Mr. C. R. Kossack, Mr. J. R. Lee, Mr. P. E. Poe, Mr. James Smillie.

The death in 1941 of Professor H. L. Lebesgue of the Collège de France is now reported.

Dean John Matheson of Queen's University died January 24, 1944, at the age of seventy-one years. He had been a member of the Society since 1918.

Professor Emeritus H. H. Dalaker of the University of Minnesota died May 20, 1943. He had been a member of the Society since 1909.

Professor Emeritus T. S. Fiske of Columbia University died January 10, 1944. Further notice of his death will appear in a forthcoming issue of the BULLETIN.

Dr. Edward Helly of the Illinois Institute of Technology died November 28, 1943.

Mr. H. L. Mintzer died November 24, 1943.

Associate Professor G. A. Pfeiffer of Columbia University died December 28, 1943. He had been a member of the Society since 1914.

Professor H. L. Rietz of the State University of Iowa died December 7, 1943. He had been a member of the Society since 1902.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

53. A. W. Jones: *The lattice isomorphisms of certain finite groups.*

Let G be any finite group whose lattice of subgroups is modular. The author gives an explicit procedure for obtaining defining relations for all groups whose lattices are isomorphic to the lattice of G. Since abelian groups have lattices which are modular, the procedure given includes a characterization, for the finite case, of the groups which are lattice isomorphic to abelian groups (cf. Baer, Amer. J. Math. vol. 61 (1939) pp. 1–44). (Received January 28, 1944.)

54. Oystein Ore: *Galois connexions.*

In numerous mathematical theories there occur order inverting correspondences between two structures P and Q, $p \to Q(p)$, $q \to P(q)$, such that $PQ(p) \supset p$, $QP(q) \supset q$. The basic properties of such Galois correspondences are derived. They correspond to a duality between structures or closure relations and can also be considered to be mappings of closure relations. They are closely connected with the theory of binary relations as discussed in the author's Colloquium Lectures given in Chicago in 1941. Among the applications should be mentioned a general Galois theory for binary relations, illustrated in detail for the case of equivalence relations. (Received December 7, 1943.)

55. Raphael Salem: *On a remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan.*

Let us denote by C the set of all algebraic integers such that all their conjugates have moduli inferior to 1 ("Pisot-Vijayaraghavan numbers" or briefly "P. V. numbers"). It is proved that the set C is closed. C being enumerable it follows that it is (1) nowhere dense; (2) not dense in itself; (3) reducible. There exists a number larger than 1 which is the smallest "P. V. number." (Received December 29, 1943.)

56. P. M. Whitman: *Identities in lattices of equivalence relations.* Preliminary report.

It is shown that free lattices can be represented by equivalence relations. Therefore no lattice identity can hold in every lattice which consists of all equivalence relations on some set, unless it holds in every lattice. (Received December 27, 1943.)

57. P. M. Whitman: *Lattices and equivalence relations.* Preliminary report.

It is shown that any lattice is isomorphic to a sublattice of the lattice of all equivalence relations on some set. (Received January 28, 1944.)

Analysis

58. Jesse Douglas: *Separable transformations with separable inverse.*

All transformations $X=f(x)+h(y)$, $Y=g(x)+k(y)$ are found whose inverses are of the same form. Six essentially different types are obtained. If x, y are interpreted as minimal coordinates $u+iv$, $u-iv$ (and X, Y similarly), we have all harmonic transformations whose inverses are harmonic. The paper will be published in full. (Received January 15, 1944.)

59. K. O. Friedrichs: *The identity of weak and strong extensions of differential operators.*

In applying the theory of linear operators in Hilbert spaces or spaces L_p to the solution of differential equation problems, it is impossible to retain the meaning of differentiation in the ordinary sense; the concept of differential operator must be extended. Two such extensions offer themselves, a "weak" and a "strong" one, that is, the adjoint of the "formal-adjoint" and essentially the closure. The purpose of the paper is to prove the identity of these two extensions for general linear differential operators. The main tool for the proof is a certain class of smoothening operators approximating unity. They yield the identity of both extensions immediately for differential operators with constant coefficients; they are a strong enough tool to yield this identity likewise for operators with non-constant coefficients. (Received December 3, 1943.)

60. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.*

Let $U(z, \bar{z})$, $z=x+iy$, $\bar{z}=x-iy$, x, y real, be a real solution of $L(U)=\Delta U+AU_x+BU_y+CU=0$, where A, B, and C are entire functions when x and y are extended to complex values. To every such solution corresponds uniquely a complex solution $u(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}z^m\bar{z}^n$ of $L(U)=0$, with the property that $\sum_{n=0}^{\infty}A_{0n}\bar{z}^n=\pi U(0,0)\exp\left(-\int_0^{\bar{z}}a(0,\bar{z})d\bar{z}\right)$, where $a(z, \bar{z})=(1/4)\{A[((z+\bar{z})/2, (z-\bar{z})/2i)]+iB[((z+\bar{z})/2, (z-\bar{z})/2i)]\}$. These solutions were introduced by Bergman (Rec. Math. (Mat. Sbornik) N. S. vol. 2 pp. 1169–1198 and Trans. Amer. Math. Soc. vol. 53 pp. 130–155) who showed that the location of the singularities of $u(z, \bar{z})$ is determined by the sequence $\{A_{m0}\}$. Employing this result the author investigates the relations between sequences $\{A_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, and the positions of singularities of $u(z, \bar{z})$. For example, using a result of Mandelbrojt (C. R. Acad. Sci. Paris, 1937, pp. 1456–1458) he determines the arguments of the singularities on the circle of convergence of $u(z, \bar{z})$ in terms of the sequence $\{A_{mk}\}$, k fixed. In the last section of the paper, using explicitly an integral representation of the complex solutions $u(z, \bar{z})$, the author investigates the real solutions $U(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}D_{mn}z^m\bar{z}^n$ of $L(U)=0$. He constructs, in terms of $\{D_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, and some of the deriva-

tives of the coefficients of L, an analytic function $f(z/2)$ whose singularities coincide with those of $U(z, \xi)$. (Received January 28, 1944.)

61. Mark Kac: *On real zeros of some functions.*

The purpose of this note is to call attention to some applications of the formula for the number of real roots given recently by the author (*On the distribution of values of trigonometric sums with linearly independent frequencies*, Amer. J. Math. vol. 65 (1943) pp. 609–615, in particular §3). If $f(x)$ has a continuous first derivative and a finite number of turning points in (a, b) then the number of zeros of $f(x)$ in (a, b) is given by the formula $(\pi/2)\int_{-\infty}^{+\infty}\{\int_a^b \cos(\xi f(x))|f'(x)|dx\}d\xi$, with the understanding that multiple roots are counted only once and that end points (a and b) if they happen to be roots are counted as 1/2 each. This formula is a source of amusing identities. For instance if $f(x)=\cos x$, $a=0$, $b=2\pi$ one obtains easily that $4\pi^2=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}((J_0(\xi)-J_0(\xi^2+\eta^2)^{1/2})/\eta^2)d\xi d\eta$. Also this formula is very convenient in computing average numbers of real roots of certain random functions. (Received January 31, 1944.)

62. A. T. Lonseth: *Dirichlet's principle for $\Delta u = F_u(u; x, y)$.*

This nonlinear elliptic partial differential equation (Δ = Laplacean operator) is the Euler-Lagrange equation for the minimization of $\iint\{u_x^2+u_y^2+2F(u; x, y)\}dx\,dy$. Here the simultaneous boundary-value and minimum problem solution is obtained by modifying Courant's method for self-adjoint linear elliptic equations (R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. II, Berlin, 1937, chap. 7). It is assumed that $F(u; x, y)$ is of class C'' and convex in u, and of class C' in x and y. Assumptions as to region and boundary values are as in Courant-Hilbert. (Received January 17, 1944.)

63. Szolem Mandelbrojt: *Some theorems connected with the theory of infinitely differentiable functions.*

The author gives a simple and elementary proof of the necessity of a well known necessary and sufficient condition for quasi analyticity, and an immediate proof of a theorem concerning Watson's problem. The author proves also an inverse to a theorem of S. Bernstein on the best approximation on the whole real axis. (Received January 27, 1944.)

64. H. E. Newell: *The solutions of a certain linear matric differential equation.*

The linear matric differential equation $dY(x, \lambda)/dx = \{\lambda(\delta_{ij}r_j(x))+(q_{ij}(x, \lambda))\} \cdot Y(x, \lambda)$, x and λ being complex, has been shown to possess under certain conditions solutions of the form $P(x, \lambda)E(x, \lambda)$, where $E(x, \lambda)=(\delta_{ij}\exp\{\lambda\int^x r_j(x)dx\})$ and $P(x, \lambda)$, analytic in x, reduces uniformly in x to the identity matrix when λ becomes infinite. The theory, originally developed by R. E. Langer (R. E. Langer, *The boundary problem of an ordinary linear differential system in the complex domain*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 151–162; correction, p. 467) and later extended by the author (H. E. Newell, Jr., *The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain*, Duke Math. J. vol. 9 (1942) pp. 245–258, and vol. 10 (1943) pp. 705–709), was applied for the most part to cases in which the elements $q_{ij}(x, \lambda)$ were analytic and bounded. The present paper treats of certain special cases in which the functions $q_{ij}(x, \lambda)$ may have poles on the boundary of the x region in question. (Received January 21, 1944.)

65. A. M. Peiser: *Some applications of Fourier analysis to the study of real roots of algebraic equations.*

Let $N_n(x) = N_n(x_0, \cdots, x_n)$ denote the number of real roots of $F = \sum_{j=0}^{n} x_j t^j = 0$. It is shown that at each continuity point of N_n, $N_n(x) = \lim_{h\to\infty}(h/(2\pi)^{1/2}) \int_{-\infty}^{\infty} A(t) \cdot |Y_n(x, t)| \exp[-h^2 X_n^2(x, t)/2]dt$ where $A(t)$ is a known positive function of t alone, and X_n and Y_n are orthogonal, normal linear forms in $x_0, \cdots, x_n$. Thus, if the x's are independent, normally distributed random variables, then X_n and Y_n are normally distributed and, because of the orthogonality, independent. For the average number (mathematical expectation = m.e.) of real roots of $F = 0$ in this case we find m.e. $\{N_n(x)\} = \lim_{h\to\infty}(h/(2\pi)^{1/2}) \int_{-\infty}^{\infty} A(t)$ m.e. $\{|Y_n(x, t)|\}$ m.e. $\{\exp[-h^2 X_n^2(x, t)/2]\}dt$. Suppose now that m.e. $\{x_j\} = c_j$ and that for each j, m.e. $\{x_j^2\} = s$. The author compares the number of real roots $N_n(c)$ of $F_1 = \sum_{j=0}^{n} c_j t^j = 0$ with the number of real roots $N_n(c+sx)$ of $F_2 = \sum_{j=0}^{n} (c_j + sx_j) t^j = 0$, and shows how to choose s so that m.e. $\{N_n(c+sx)\}$ is near $N_n(c)$. This may have an application to empirical equations, for we can consider $F_1 = 0$ as the theoretical, or "true" equation, and $F_2 = 0$ as an "observed" equation in which each coefficient is measured with precision s. (Received January 28, 1944.)

66. Harry Pollard: *Fourier series with coefficients in a Banach space.*

Let $f(t)$ be a function on (0, 1) to the complex Banach space B. Bochner has shown that a large part of the older theory of Fourier series carries over to functions of this character, but breaks down in the fundamental L^2 theory. In particular Bessel's inequality is no longer valid. It is proved in this paper that the inequality carries over if and only if B admits a scalar product. (Received January 24, 1944.)

67. C. E. Rickart: *Representation of linear transformations on summable functions.*

This paper contains an integral representation theorem for the general bounded linear transformation on summable functions to an arbitrary Banach space. The representation is obtained by means of an abstract Radon-Nikodym theorem proved earlier by the author (see abstract 49-11-270). (Received January 26, 1944.)

68. H. E. Robbins: *A note on the Riemann integral.*

Let $f(x)$ be a continuous function. If, in the usual definition of the Riemann integral of $f(x)$ from a to b as the limit of S_n equals the sum of terms of the form $f(\xi_i)(x_i - x_{i-1})$, the points of subdivision, $x_0 = a, x_1, \cdots, x_n = b$, are *not* assumed to have the property $x_i \geqq x_{i-1}$, then S_n need not tend to a limit as $d_n = \max|x_i - x_{i-1}|$ tends to 0. But if any constant $C \geqq (b-a)$ is given in advance and if the points of every subdivision satisfy the inequality, sum of $|x_i - x_{i-1}| \leqq C$, then S_n will tend to the Riemann integral of $f(x)$ as d_n tends to 0. (Received January 28, 1944.)

69. F. H. Safford: *Analysis of a non-harmonic wave.*

The present paper considers the problem of expressing a periodic function by means of sines and cosines up to p harmonics, when the values of the function are given as ordinates for equidistant abscissas, with no further information. This is a variation of the method of selected ordinates used by J. Fischer-Hinnen in 1901, but

with p coefficients for the sine terms and $p+1$ coefficients for the cosine terms, corresponding terms being combined into the form $A \sin n(t+\theta)$. The period may be taken as 2π, and the values of n are from 0 to p. It is necessary to use a second set of ordinates at the distance $(\pi/2p)$ from the first set thus providing $2p+1$ disposable constants corresponding to the same number of points from the observed function. The presence of higher harmonics with composite indices is duly provided for. The theorem concerning the sums of equidistant ordinates of sine curves provides the components of the observed ordinates and the latter are linearly related to the disposable constants. Thus for each harmonic it is possible to obtain its amplitude and phase angle without trigonometric ambiguity, in terms of the observed values of the given ordinates. (Received January 24, 1944.)

70. Raphael Salem: *On a theorem of Bohr and Pál.*

The paper gives a short and simple proof of the following extension of a theorem of Pál, due to H. Bohr: given a function $\phi(t)$ continuous and of period 2π, there exists a function $t(\theta)$ $(t(0)=0,\ t(2\pi)=2\pi)$, continuous, strictly increasing, such that the Fourier series of $\phi(t(\theta))$ converges uniformly for $0 \leq \theta \leq 2\pi$. (Received December 29, 1943.)

71. Raphael Salem: *Sets of uniqueness and sets of multiplicity.* II.

This paper is a continuation of a paper which appeared under the same title in Trans. Amer. Math. Soc. vol. 54 (1943) pp. 218–228. It gives the necessary and sufficient condition for an unsymmetric perfect set of constant ratio to be a set of uniqueness. It deals also with some aspects of the classification in sets of uniqueness and sets of multiplicity of symmetrical perfect sets of the Cantor type and of variable ratio of dissection. (Received December 29, 1943.)

72. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* IV.

Whenever an iterative compositional functional equation has a solution containing an arbitrary function in two or more variables, the possibility of an associated representation of substitution groups by functional equations may be entertained. A simple example is $f(u, x_2, \cdots, x_n, v)=f(x_1, x_2, \cdots, x_{n+1})$ where $u=f(x_1, i_1, i_2, \cdots, i_n)$ and $v=f(x_{n+1}, i_1, i_2, \cdots, i_n)(n=1, 2, \cdots)$ with solution: $f(x_1, x_2, \cdots, x_{n+1})=x_1-x_{n+1}+\alpha(x_2, \cdots, x_n)$. A group of equations follows if $f(x_1, x_2, \cdots, x_{n+1})=f(x_1, x_{i_2}, \cdots, x_{i_n}, x_{n+1})$. In this and certain analogous representations of a substitution group G, the equation corresponding to the identical substitution has a solution containing an arbitrary function and the corresponding group of equations has a solution based on the invariance of the latter function under G. A generalized associative equation as inverse of the above equation corresponds to the identical element in the group of equations: $\phi\{\phi(x_1, x_2, \cdots, x_n, y_1), y_2, \cdots, y_n, y_{n+1}\}=\phi\{x_1, x_2, \cdots, x_n, \phi(y_1, y_2, \cdots, y_n, y_{n+1})\}$. Another example is based on $f(u, u, \cdots, u)=u$, where $u=f(x_1, x_2, \cdots, x_{n+1})$ with solution: $f(x_1, x_2, \cdots, x_{n+1})=\alpha(w_1, w_2, \cdots, w_n)+x_{n+1}$ where $w_i=x_i-x_{n+1}$ and α is arbitrary except for $\alpha(0, 0, \cdots, 0)=0$. (Received January 24, 1944.)

73. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* V.

The quasi-transitive equation, $f(u_1, u_2, \cdots, u_{n+1})=f(x_1, x_2, \cdots, x_{n+1})$,

$u_i = f(x_i, i_1, i_2, \cdots, i_n)$, is made the basis of an $(n+1)$-adic genesis of the group concept by postulating further: (1) Given $f(x_1, x_2, \cdots, x_{n+1})$, there exists $\phi(x_1, x_2, \cdots, x_{n+1})$ such that $f\{\phi(x, i_1, i_2, \cdots, i_n), i_1, i_2, \cdots, i_n\} = x$ and $\phi\{f(x, i_1, i_2, \cdots, i_n), i_1, i_2, \cdots, i_n\} = x$. (2) If $x \neq y$, then $f(x, x, \cdots, x) = f(y, y, \cdots, y)$. (3) The set S of elements x_i is closed under the compositions f and ϕ. The preceding postulates define a group under binary composition if in the functions $f(x_1, x_2, \cdots, x_{n+1})$ and $\phi(x_1, x_2, \cdots, x_{n+1})$, $x_2 = x_3 = \cdots = x_{n+1}$. The postulates are also satisfied in the domain of abstract groups under $(n+1)$-ary composition if $f(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot x_{n+1}^{-1} \cdot \alpha(w_2, w_3, \cdots, w_n)$ where $w_i = x_i \cdot x_{n+1}^{-1}$ and α is an arbitrary "function" of group elements. Finally, the postulates are satisfied by a set of elements closed under the binary compositions of a commutative group and any one of a class of specialized operations represented by the function α, including multiplication of arguments. An analogous genesis based on a generalized associative equation is discussed. (Received January 24, 1944.)

74. M. F. Smiley: *An extension of metric distributive lattices with an application in general analysis.*

It is shown that a metric distributive lattice (Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, 1940, pp. 41 and 74) may be embedded in a field of sets with a finitely additive measure provided that the modular functional of the basic lattice is bounded from below. Under the additional hypothesis that this functional is bounded, it is proved that the corresponding measure is *completely additive*. This result is applied to E. H. Moore's last instance of his second general analysis theory (*General analysis*, Part I, Memoirs of the American Philosophical Society, vol. 1, 1935, pp. 14–15). In this example the positive character of Moore's basic matrix is found to depend essentially on the distributive law. The proof of the major result employs theorems of H. Wallman (*Lattices and topological spaces*, Ann. of Math. (2) vol. 39 (1938) pp. 112–126) and of M. H. Stone (*Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 376–481) on the representation of Boolean rings, as well as methods of H. M. MacNeille (*Extension of a distributive lattice to a Boolean ring*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 452–455) and of S. Kakutani (*Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. (2) vol. 42 (1941) p. 533). (Received January 10, 1944.)

75. F. T. Wang: *On Riesz summability of Fourier series.* IV.

Let $f(t)$ be an integrable periodic function with period 2π and $\phi_\alpha(t)$ be the fractional integral of $\phi(t) = 2^{-1}\{f(x+t) + f(x-t) - 2s\}$ of order α. Then we have the following results: (1) If $\phi_\alpha(t) = o(t^\alpha / \log t^{-1})$, $\alpha > 0$, as $t \to 0$, then the Fourier series of $f(t)$ is summable by Riesz typical mean of type $\lambda_n = \exp(\log n)^{1+\alpha^{-1}}$ and of order $\alpha+1$, or simply, summable $\exp(\log n)^{1+\alpha^{-1}}$, $\alpha+1$, to sum s at $t=x$. (2) If $\sum_{n=2}^{\infty} (a_n^2 + b_n^2)(\log n)^\alpha$ $(0 < \alpha < 1)$ converges then the series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is summable $\exp(\log n)^{\alpha+1}$, δ, $\delta > 0$, almost everywhere. (Received January 5, 1944.)

76. F. T. Wang: *Some remarks on oscillating series.*

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$ where r is a positive integer. Then we have the following result: If $\sigma_n^{(r)} - s = o(n^{-r\alpha})$, $0 < \alpha < 1$, as $n \to \infty$ and $a_n > -Kn^{\alpha-1}$, then the series $\sum_{n=0}^{\infty} a_n$ converges to sum s. For the case $r=1$ this has been proved by Boas. The author proves in this note the result for $r=2$, and gives an

example from the theory of Fourier series that there exists a divergent series $\sum_{n=0}^{\infty} a_n$ such that $\sigma_n^{(2)} - s = 0(n^{-2\alpha'})$, $0 < \alpha' < \alpha < 1$ and $a_n = O(n^{\alpha-1})$. (Received January 5, 1944.)

77. D. V. Widder: *The iterates of the Laplace kernel.*

The iterates of the Laplace kernel $G_0(x, y) = e^{-xy}$ are defined by the recursion relation $G_n(x, y) = \int_0^\infty G_0(x, t) G_{n-1}(t, y) dt$, $n = 1, 2, \cdots$. In an earlier paper (Bull. Amer. Soc. vol. 43 p. 813) the author determined explicitly all these functions which have odd subscripts. They are rational functions of x, y and $\log(x/y)$. In the present paper the iterates with even subscripts are studied. They cannot be expressed in terms of the elementary functions. Complete asymptotic series for their behavior near $x = +\infty$ are obtained. For their behavior near the origin they are developed in power series which converge for all positive values of x. The method consists in expressing the nth iterate in terms of a function $h_n(x)$ whose bilateral Laplace transform turns out to be $\Gamma(-s)^{n+1}\,\Gamma(s+1)^n$. The function $h_n(x)$ is in turn expressible in terms of a new set of transcendental functions related to the familiar exponential integral $EI(x)$. (Received January 14, 1944.)

Applied Mathematics

78. Stefan Bergman: *On solutions of certain partial differential equations in three variables.*

(I) Let $\mathbf{E}(x, y, z, t, \tau)$ be a solution of the equation $G(\mathbf{E}) = L(\mathbf{E}) + [\tau^{-1}U^{-1}(1-\tau^2)(\mathbf{E}_z + i\mathbf{E}_y \cos t + i\mathbf{E}_x \sin t)_\tau] + A[(1/2)\tau^{-1}(1-\tau^2)\mathbf{E}]_\tau$ which satisfies certain boundary conditions. Here $L(\mathbf{E}) = \Delta\mathbf{E} + A(x\mathbf{E}_x + y\mathbf{E}_y + z\mathbf{E}_z) + C\mathbf{E}$, $U = x + iy \cos t + iz \sin t$. Then $\psi(x, y, z) = \int_0^{2\pi} \int_{-1}^{1} \mathbf{E} f[(1/2)u(1-\tau^2), t]/dt d\tau$, where $f(\zeta, t)$ is an arbitrary analytic function of ζ and t, will be a solution of $L(\psi) = 0$. (II) If A and C are entire functions of $r^2 = x^2 + y^2 + z^2$ alone, then $G(\mathbf{E})$ becomes $G_0(\mathbf{E}) = (1-\tau^2)\mathbf{E}_{\tau\tau} - \tau^{-1}(1+\tau^2)\mathbf{E}_\tau + r\tau[\mathbf{E}_{\tau\tau} + 2r^{-1}\mathbf{E}_\tau + B\mathbf{E}] = 0$ where $B = [-(3/2)A - (1/2)rA_r + C - (1/4)r^2A^2]$. The author shows that in the case II, there always exists a solution $\mathbf{E} = \mathbf{H}(r, \tau)$ of $G_0(\mathbf{E}) = 0$, which is an entire function of r. The author considers vectors $S = (\psi_1, \psi_2, \psi_3)$ where $\psi_1 = \int_0^{2\pi} \int_{-1}^{1} \mathbf{H} f dt d\tau$, $\psi_2 = i\int_0^{2\pi} \int_{-1}^{1} \mathbf{H} f \cos t dt d\tau$, $\psi_3 = i\int_0^{2\pi} \int_{-1}^{1} \mathbf{H} f \sin t dt d\tau$. Clearly $L(\psi_K) = 0$, $K = 1, 2, 3$. Let c^1 be a simple closed curve which lies on a sphere $x^2 + y^2 + z^2 = \text{const.}$ If the ψ_K are regular in this sphere then $\int_{c^1} S \cdot dX = 0$. Here $X = (x, y, z)$, and $\cdot$ means the interior product. "Residue" theorems are derived if the ψ_K have singularities in the above sphere. Applications in the theory of waves propagation are indicated. (Received January 28, 1944.)

79. Nathaniel Coburn: *A boundary value problem in plane plasticity for the Coulomb yield condition.*

The following problem is studied: given a semi-plane $x > 0$, composed of plastic material which follows the Coulomb yield condition; the stresses σ_x, σ_y, σ_{xy} acting on the boundary $x = 0$ are considered as known; to find the stresses at any point in the interior of the semi-plane. The method of attack is a modification of that used in studying a similar problem for a perfectly plastic material. The stresses and the functions $\sin 2\gamma$, $\cos 2\gamma$ (where γ is the angle between the x-axis and a tangent to a line of principal shearing stress) are expanded in power series of the friction coefficient. Substituting these power series into the Levy equations, there results an infinite set of Levy equations for the various approximations to the stresses. By requiring that

each set of approximations satisfy the equilibrium relations, an infinite sequence of Airy stress functions is obtained. Since each Airy stress function must satisfy two independent Levy equations, there results an infinite set of second order linear partial differential equations. The various approximations are determined and the convergence of σ_{xy} is demonstrated for the case $\sigma_{xy}(0, y) = 0$. (Received December 20, 1943.)

80. H. L. Garabedian: *The analogue of Bromwich's theorem for integral transformations.*

The main theorem of this paper provides sufficient conditions in order that a method of summation defined by an integral transformation of the type $s(x) = \int_0^\infty k(x, t)a(t)dt$ shall include Cesàro summability of positive integral order n, defined by the transformation $y(x) = \int_0^\infty (1 - t/x)^n a(t)dt$. A kernel $k(x, t)$ is exhibited which fulfills the conditions of this theorem and thus provides a means of solving in a rigorous fashion the problem of the flow of heat in an infinite rod whose surface is a non-conductor. (Received December 20, 1943.)

81. Michael Goldberg: *A three-space analog of a plane Kempe linkage.*

The following theorem is due to A. B. Kempe: Each of the sides of a plane quadrilateral linkage may be joined by a link to a common pivot without restricting the movability of the linkage. The new pivot associated with a side need not be collinear with the other pivots of that side; the three pivots may form a triangle in a rigid plate. Extensions and further studies on this theorem were made by Kempe (*On conjugate four-bar linkages*, Proc. London Math. Soc. vol. 9 (1878) pp. 133–147) and Darboux (*Recherches sur un système articulé*, Bull. Sci. Math. vol. 3 (1879) pp. 151–192). This paper is concerned with the analogous theorem in three-space. It is shown that it is possible for each of the four bodies of a movable hinged skew quadrilateral (a Bennett linkage) to be joined by a hinged link to a common hinge without restricting the movability. As in the case of the plane, the three hinges in a body need not have a common perpendicular. In general, these hinge lines are non-parallel and non-intersecting. (Received January 29, 1944.)

82. Max Herzberger: *The diapoint characteristic.*

To every point in an optical system with symmetry axis belongs a manifold of *diapoints*, the intersections of the image rays with the meridian plane. If the object point has a sharp image, all the diapoints coincide with it; if the object point has an image with a curved or straight symmetry axis, the diapoints form the axis. The author introduces the diapoint characteristic, a function giving the optical path between point and diapoint. Such a function has a two-dimensional manifold of values in the general case, a one-dimensional manifold in case of symmetry, and is constant if the object point is sharply imaged. The diapoint function $F(x, y, z, v, w)$ can be constructed as a function of the coordinates of the object point (x, y, z) and two parameters v, w. Its analysis for a given system gives at once all the points in space which are sharply or symmetrically imaged, and tells whether there are any surfaces sharply or symmetrically imaged. An example, refraction at a plane and at a sphere, is analyzed. (Received January 24, 1944.)

83. R. P. Isaacs: *Airfoil theory for flows of variable velocity.*

There arise many cases in aircraft practise—conspicuously that of the rotating

wing—where the flow over the lift surface is of variable velocity. The first section of this paper deals with theory of an airfoil in a variable speed stream (but constant angle of attack). The result is an integrodifferential relation between the two functions involved: velocity and circulation. Next the case is treated: velocity = constant + sinusoidal variation. Finally the lift is studied and given explicitly for this latter case in the form of a Fourier series. (Received January 24, 1944.)

84. R. v. Mises: *Note on the numerical solution of linear partial differential equations.*

The usual method of numerical solution consists in replacing the differential equation $L(u)=0$ by a set of algebraic linear equations for the values $u(P_i)$ where P_i are the points of a rectangular lattice. This procedure has obvious inconveniences in the case of curved boundaries. Many questions arise if instead of this lattice more general types of point sets are used. The present paper deals with some of these questions, in particular with the number and the arrangement of points required for any order of approximation and with the possibility of applying the method of successive approximations (relaxation method) to the solution of the algebraic equations. Specified formulae are proposed for the case of the Laplace equation. (Received January 28, 1944.)

85. David Moskovitz: *Numerical solution of Laplace's and Poisson's equations.*

This paper considers the system of equations (1) $L_x u_j(i) - u_{j+1}(i) - u_{j-1}(i) = \phi_j(i)$ $(i=1, 2, \cdots, n-1; j=1, 2, \cdots, m-1)$, where L_x is the linear operator defined by $L_x f(i) = cf(i) - f(i+1) - f(i-1)$; c is a constant, the $\phi_j(i)$ are prescribed, and we seek the values $u_j(i)$. The solution of (1) is obtained in symbolic form which is interpreted to define actual solutions. Application is made to the numerical solution of Laplace's and Poisson's equations with prescribed values on a rectangular boundary. Tables of values are included to facilitate making the applications. (Received January 28, 1944.)

86. Isaac Opatowski: *Use of special functions in a problem of uniform strength.*

A cantilever is bent under the action of its own weight and a concentrated load F at the free end. If x is the distance of the cantilever's cross section from its free end, $A(x)$ the area of the cross section, $S(x)$ the section modulus, g the specific weight of the beam, s the maximum stress of each cross section, the condition of uniform strength is: (*) $Fx + g\int_0^x (x-q)A(q)dq = sS(x)$. If the surface bounding the cantilever is representable in the parametric form: $y = u(x)U(t)$, $z = v(x)V(t)$ where (x, y, z) are orthogonal Cartesian coordinates, with z-axis lying in the plane of the bending couple, the use of the functions $w(x) = u(x)[v(x)]^2$, $r(x) = 1/v(x)$ (that is, w directly proportional to the section modulus, r inversely proportional to the radius of gyration) reduces (*) to the differential equation (**) $w''(x) = ar(x)w(x)$, where a is a constant depending on the type of cross section. (**) gives a wide possibility of use of most of the special functions. Cases leading to hyperbolic and cylindrical functions are discussed in detail. The treatment is general, that is, independent of the type of cross section. (Received January 28, 1944.)

87. H. E. Salzer: *Table of coefficients for inverse interpolation with advancing differences.*

This table can be used in place of the similar one described in a previous abstract (49-9-224). In addition it has the advantage that it can be employed for inverse interpolation near the beginning or end of a table and also when only a few tabulated values are available (as is the case, for instance, when solving transcendental equations). The Mathematical Tables Project has computed the coefficients of the products of ratios of advancing differences of various order. These coefficients occur in the formula obtained by the inversion of the Gregory-Newton formula for direct interpolation, employing Lagrange's theorem. The polynomial expressions for those coefficients are given in H. T. Davis, *Tables of the higher mathematical functions*, vol. 1, pp. 80–81. (A slight addition to the formula was made to complete it as far as the eighth order.) The coefficients of the two fourth order and the two fifth order terms were calculated to ten decimals, at intervals of 0.001 of the argument $m=(u-u_0)/(u_1-u_0)$. The coefficients of the four sixth order terms were calculated at intervals of 0.01 and the four seventh order coefficients as well as the seven eighth order coefficients were computed at intervals of 0.1 (all to ten decimals). (Received December 2, 1943.)

88. Andrew Vazsonyi: *On two-dimensional rotational gasflows.*

The differential equation of an inviscous compressible fluid is determined under the condition that the conductivity of the gas is negligible. (By admitting discontinuities this includes flows with shock waves.) The equation of motion is (1) $\psi_{xx}(1-(u^2/a^2))-(2uv/a^2)\psi_{xy}+\psi_{yy}(1-(v^2/a^2))=\rho^2[\partial h_0/\partial\psi-((k-1)/kR)(h_0+q^2/2)\partial s/\partial\psi]$, $u=(1/\rho)\psi_y$, $v=-(1/\rho)\psi_x$, $\rho=\epsilon^{-s/R}(h_0-q^2/2)^{1/k-1}$, $q^2(h_0-q^2/2)^{2/k-1}=\epsilon^{2s/R}(\psi_x^2+\psi_y^2)$, $a^2=k\epsilon^{-(k-1)s/R}\cdot(h_0-q^2/2)$ where the notations are as follows: ψ streamfunction, q velocity, u and v velocity components, ρ density, a local speed of sound, h_0 stagnation enthalpy (Bernoulli constant), s specific entropy, R gas constant, k isentropic exponent. There are two arbitrary functions in this equation, namely: $h_0(\psi)$ and $s(\psi)$; these must be given by the boundary conditions (or by the nature of the discontinuities). The flow is rotational in general and the rotation is given by $\omega=\partial v/\partial x-\partial u/\partial y=-\rho\partial h_0/\partial\psi+(p/R)\partial s/\partial\psi$. For irrotational flow the right-hand side of (1) equals 0. (Received January 28, 1944.)

Geometry

89. Stefan Bergman: *A Hermitian metric and its property.* Preliminary report.

Let the real analytic function $\Phi(z_1, z_2, \bar{z}_1, \bar{z}_2)$, $z_k=x_k+iy_k$, $\bar{z}_k=x_k-iy_k$, $k=1, 2$, of four real variables x_1, x_2, y_1, y_2, satisfy the equation $\Phi=c\{[\partial^2\Psi/\partial z_1\partial\bar{z}_1][\partial^2\Psi/\partial z_2\partial\bar{z}_2]-|\partial^2\Psi/\partial z_1\partial\bar{z}_2|^2\}$, $c=$constant, $\Psi=\log\Phi$, in a domain B of the (four-dimensional) space and become infinite on the boundary of B. The expression $ds_B^2(z_1, z_2)=\sum_{m,n=1}^{2}[\partial^2\Psi/\partial z_m\partial\bar{z}_n]dz_m d\bar{z}_n$ defines in B a Hermitian metric which is invariant with respect to transformations by pairs of analytic functions, $z_k^*=z_k^*(z_1, z_2)$, $k=1, 2$, of two complex variables which are regular in B. Using the methods of the theory of orthogonal functions (see Bergman, *Sur les fonctions orthogonales de plusiers variables complex avec les applications à la théorie des fonctions analytiques*, Interscience Pub-

lishers, New York, 1941, chap. 4) the author investigates the property of this metric in the large. The author introduces the system of orthogonal functions, $\{M^1(z_1, z_2), M^{21}(z_1, z_2), \cdots\}$ (see loc. cit. p. 28) which he obtains by replacing in the formula (9) (see loc. cit. p. 46) the kernel function K by Φ. Various applications to the theory of pseudo-conformal transformations are discussed. (Received December 15, 1943.)

90. S. S. Chern: *A simple intrinsic proof of the Gauss-Bonnet formula for Riemannian polyhedra.*

In an n-dimensional Riemannian space let R_{ijkl} be the Riemann-Christoffel tensor. Let $\epsilon^{i_1 \cdots i_n}$ equal 1 or -1, according as the indices form an even or odd permutation of $1, \cdots, n$, and be otherwise zero. Define $S = \epsilon^{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{n-1} i_n j_{n-1} j_n}$ when n is even and $S = 0$ when n is odd. Let the Riemannian space be orientable and closed. The Gauss-Bonnet formula in its simplest form asserts that the integral of S over the volume element is equal to a constant multiple of the Euler characteristic of the space. (Cf. C. B. Allendoerfer and André Weil, Trans. Amer. Math. Soc. vol. 53 pp. 101–129.) A simple intrinsic proof of this formula is given, based on the following two facts: (a) the integrand in question is a null form in the $(2n-1)$-dimensional space formed by the unit vectors of the Riemannian space; (b) a field of unit vectors can be defined in the space with a finite number of singular points. Of these two facts (a) is proved by an explicit formula, while (b) is well known. An application of Stokes' formula in the space of unit vectors immediately yields the desired result. Incidentally the index theorem on vector fields follows as a consequence of the present proof. Suitable modifications give the formula for a Riemannian polyhedron. Under reasonable assumptions the formula holds for an infinite Riemannian space. (Received December 13, 1943.)

91. J. M. Feld: *On a representation in space of groups of circle and turbine transformations in the plane.*

In a previous paper (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 783–790) the author showed that the oriented lineal elements of the euclidean plane can be mapped continuously and 1-1 upon the points of quasi-elliptic three-space, Q_3, so that the group of whirl-similitudes in the plane is isomorphic with the group of automorphic projectivities of Q_3. In this paper the mapping is extended to a 1-1 continuous representation of the lineal elements of the Moebius plane upon the points of projective S_3. The Laguerre, Moebius, and Lie groups of circle transformations are then shown to be isomorphic with groups of projective transformations in S_3 which leave certain linear complexes and congruences of lines invariant, and Kasner's group of turbine transformations is shown to be isomorphic with the projective group of S_3. (Received January 29, 1944.)

92. C. C. Hsiung: *Some invariants of certain pairs of hypersurfaces.*

The purpose of this paper is to generalize the results of Buzano (Bollettino della Unione Matematica Italiana vol. 14 (1935) pp. 93–98), Bompiani (Bollettino della Unione Matematica Italiana vol. 14 (1935) pp. 237–243), and the author (*Projective invariants of a pair of surfaces*, to appear in Amer. J. Math.). Two pairs of hypersurfaces V_{n-1}, V_{n-1}^* in a space S_n of n (≥ 3) dimensions are considered: (a) The tangent hyperplanes t_{n-1}, t_{n-1}^* at two ordinary points O, O^* are coincident. (b) The tangent hyperplanes t_{n-1}, t_{n-1}^* at two ordinary points O, O^* are distinct, and the common tangent flat space t_{n-2} of t_{n-1}, t_{n-1}^* contains the line OO^*. It is shown that determined by the

neighborhoods of the second order of the hypersurfaces V_{n-1}, V^*_{n-1} at the points O, O^* there exist a unique and two projective invariants for the two cases respectively; and each invariant is given metrically as well as projectively geometric characterizations. (Received December 29, 1943.)

93. G. B. Huff: *The completion of a theorem of Kantor.*

Let a complete and regular linear system Σ be defined by its order x_0 and its multiplicities $x_1, \cdots, x_\rho$ at a set, P^3_ρ, of ρ general points in the plane. A planar Cremona transformation C whose fundamental points fall at P^3_ρ will send Σ into a system Σ' of order x'_0 and multiplicities $x'_1, \cdots, x'_\rho$ at the fundamental points of C^{-1}. The characteristic $\{x\} = \{x_0; x_1, \cdots, x_\rho\}$ of Σ is related to $\{x'\} = \{x'_0; x'_1, \cdots, x'_\rho\}$ by a linear transformation L which has integral coefficients and the invariant forms $(xx) = x_1^2 + \cdots + x_\rho^2 - x_0^2$ and $(lx) = x_1 + \cdots + x_\rho - 3x_0$. In his prize memoir of 1884, S. Kantor incorrectly asserted the following proposition: any L with integral coefficients, leaving (xx), (lx) invariant, and having the coefficients of x_0 positive and all others negative, must be the description of some planar Cremona transformation. The author approaches the problem by obtaining further properties of an L associated with a C and proves the theorem: *An L is the description of some C if and only if the coefficients are integers, (xx), (lx) are invariants, and the sign of x_0 in all P-characteristics is unchanged.* Some results are listed. (Received January 26, 1944.)

94. Edward Kasner and John DeCicco: *Scale curves in cartography.*

In a non-conformal map T of a surface upon a plane, the scale function $\sigma = ds/dS$ depends upon the lineal-element (x, y, y'). The scale curves defined by $\sigma(x, y, y') = \text{const.}$ are ∞^2 in number. If the osculating circles are constructed to the ∞^1 scales passing through a fixed point P, at P, then the locus of the centers of curvature is a general cubic curve which has a node at P, the directions of the tangent lines at P coinciding with the characteristic directions. A complete analytic characterization of the ∞^2 scales is obtained. A new class of surfaces is found whose ∞^2 scales are all straight. In this class, the only surfaces of constant curvature are the developables. If the ∞^2 scales are of the Lie-Liouville cubic type, then they must form a velocity system. The cubical locus degenerates into a straight line. The velocity systems of scales define a new class of surfaces which are studied in detail. (Received January 25, 1944.)

95. Edward Kasner and John DeCicco: *Scale curves in conformal maps.*

In any conformal map Γ of a surface upon a plane, the scale function $\sigma = ds/dS$ depends upon (x, y) only. Thus there are ∞^1 scale curves $\sigma(x, y) = \text{const.}$ which may be any simple family. If it be demanded that an isothermal (or parallel) system be scale curves of a conformal map Γ such that the gaussian curvature is constant along the scales, then the surface must be developable (Δ) or applicable upon a surface of revolution (Σ). For (Δ), the scales can be any isothermal family, whereas for (Σ), the scales are concentric circles or parallel lines. If the scales are straight or circular such that the gaussian curvature is constant along them, then the surface is a (Δ) or a (Σ), and the scales must be pencils of lines or circles. The application of these results

to any conformal map of a sphere upon a plane lead to new characterizations of the Mercator, stereographic, and the general Lambert conical projections. Thus the only conformal map with straight scale curves is the Mercator; and the only circular cases are the stereographic and Lambert maps. (Received January 25, 1944.)

Statistics and Probability

96. Benjamin Epstein and C. W. Churchman: *On the statistics of sensitivity data.*

"Sensitivity data" is a general term for that type of experimental data for which the measurement at any point in the scale destroys the sample. The paper is a generalization of a method of treating such data due to Spearman. (C. Spearman, *The method of "right and wrong cases" (constant stimuli) without Gauss' formulae*, British Journal of Psychology vol. 2 (1908) pp. 227–242.) Formulae for the moments and their standard sampling errors are given. Certain minimization problems are also discussed. (Received January 26, 1944.)

97. E. J. Gumbel: *The observed return period.*

The theoretical return period $T(x)$ of a value equal to, or greater than, x is defined as the inverse of the probability $1-F(x)$. The question is how to calculate, for n observations, the return period $T(x_m)$ of the mth observation $x_m (m=1, 2, \cdots, n)$, and especially $T(x_n)$ of the largest observation x_n for an unlimited variate. This problem is important in probability papers where the variate is plotted as a function of the return period. Engineers use a compromise between the exceedance interval $'T(x_m) = n/(n-m)$ and the recurrence interval $''T(x_m) = n/(n-m+1)$, namely $T(x_m) = n/(n-m+1/2)$. In this case $T(x_n) = 2n$. If, however, the probability $F(\mathfrak{x}_n)$ of the median $\mathfrak{x}_n$ of the largest value is attributed to x_n, $T(x_n) = 1.44n+1/2$. Both methods can hardly be justified. The author attributes the probability $F(\tilde{x}_n)$ of the most probable largest value $\tilde{x}_n$ to x_n. Then $T(x_n)$, as is to be expected, converges toward n, and equals n for the exponential distribution, and $n+1$ for the logistic distribution. In the same way, the probability $F(\tilde{x}_1)$ of the most probable smallest value $\tilde{x}_1$ is used, for an unlimited variate, as frequency of the smallest observation x_1. The frequencies $F(\hat{x}_m)$ of the intermediate $n-2$ observations are obtained by linear interpolation between $F(\tilde{x}_1)$ and $F(\tilde{x}_n)$. Thus the return periods may be determined for all observations. (Received January 27, 1944.)

98. H. E. Robbins: *On the measure of a random set.*

Let X, a measurable subset of Euclidean n-dimensional space E, be a random variable (for example, X may be the set-theoretical sum of N possibly overlapping and independently chosen unit intervals on a line with a given probability distribution for their centers). Let $m(X)$ denote the measure of X, and for any point x of E let $p(x)$ denote the probability that X contain x. Then under very general hypotheses on X it is shown that the expected value of $m(X)$ is equal to the integral over E of $p(x)$. More generally, the expected value of the rth power of $m(X)$ is equal to the integral over rn-dimensional space of the function $p(x_1, \cdots, x_r)$ = probability that X contain all the points $x_1, \cdots, x_r$. (Received January 28, 1944.)

TOPOLOGY

99. C. L. Clark: *Arc reversing transformations.*

If A and B are separable metric spaces, B nondegenerate, a single-valued continuous transformation $f(A)=B$ is said to be arc reversing provided the inverse of every simple arc in B is a simple arc in A. After several basic results are obtained, it is shown that if $f(A)=B$ is arc reversing, with A locally compact and B a locally connected generalized continuum, then the sets $A-A_1$ and $B-B_1$ are homeomorphic, where B_1 is the set of all points b in B whose inverses $f^{-1}(b)$ are nondegenerate and $A_1=f^{-1}(B_1)$. In particular if $f(A)=B$ is (1-1), A and B are homeomorphic. A characterization of arc reversing transformations is afforded by the result that a single-valued continuous transformation $T(A)=B$, where A and B are locally connected continua, is arc reversing if and only if the set of inverses $[T^{-1}(b)]$, b in B, consists of single points and at most a countable number of free arcs whose end points are of Urysohn-Menger order at most 2 in A. Further results are obtained concerning local separating points and continua having homeomorphs of finite linear measure. (Received December 29, 1943.)

100. Mariano García: *Component orbits under pointwise recurrent homeomorphisms.*

A point x of a separable metric space X on which a homeomorphism $f(X)=X$ is defined is called recurrent under f if, given any neighborhood U of x, there exists an N such that $f^N(x)\in U$, and an invariant set L in X whose components can be ordered in a sequence $\cdots, A_{-2}, A_{-1}, A_0, A_1, A_2, \cdots$ such that $f(A_i)=A_{i+1}$ is defined as a component orbit. Using methods analogous to those used by Whyburn in proving the results that Hall and Schweigert obtained relative to periodic component orbits (component orbits having a finite number of components) under a pointwise periodic homeomorphism on X, this paper establishes extensions of these results to non-pointwise periodic mappings. It is shown for example that if $f(X)=X$ is a homeomorphism on a compact space X and $G_1, G_2, \cdots$ is a sequence of component orbits whose limit inferior contains a periodic component orbit Q, and if (either) each point of $\limsup G_i - Q$ is recurrent under both f and f^{-1} or each point of $\sum_{i=1}^{\infty} G_i + \limsup G_i$ is recurrent under f, then $\limsup G_i$ is a periodic component orbit. (Received December 27, 1943.)

101. W. H. Gottschalk: *Powers of homeomorphisms with almost periodic properties.*

Let X be a topological space and let $f(X)=X$ be a homeomorphism. A point x of X is said to be *recurrent* under f provided that to each neighborhood U of x there corresponds a positive integer n such that $f^n(x)\in U$. A point x of X is said to be *almost periodic* under f provided that to each neighborhood U of x there corresponds a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers with the properties that the numbers $n_{i+1}-n_i$ $(i=1, 2, \cdots)$ are uniformly bounded and $f^{n_i}(x)\in U$ $(i=1, 2, \cdots)$. A subset Y of X is said to be *minimal* under f provided that Y is nonvacuous, closed and invariant under f, and furthermore Y does not contain a proper subset with these properties. The following theorems are established: (1) If $x\in X$ is recurrent under f, then x is also recurrent under f^n for every positive integer n. (2) If X is minimal under f but not under f^k, where k is a nonzero integer, then there exists an integer n, $n>1$, such that n divides $|k|$ and f^n gives a finite minimal-set decomposition of X which

contains exactly n elements. These theorems are developed and applied to yield further results, among them the following: If X is a compact metric space and if $x \in X$ is almost periodic under f, then x is also almost periodic under f^n for every integer n. (Received December 31, 1943.)

102. D. W. Hall: *On rotation groups of plane continuous curves under pointwise periodic homeomorphisms.*

This note makes use of the work of G. T. Whyburn on light interior transformations and orbit decompositions of certain spaces to obtain a theorem by means of which a certain subset of the orbits of points under a pointwise periodic transformation may be given a linear ordering. This theorem is then used to obtain an accessibility theorem for plane continuous curves similar to one previously published by L. Whyburn (Fund. Math. vol. 28 (1937) p. 127), but having the emphasis in the hypothesis placed upon the type of the transformation rather than upon the order of the rotation group under consideration. (Received December 18, 1943.)

NEW PUBLICATIONS

BLAND, J. R. See KELLS, L. M.

CAMM, F. J. A refresher course in mathematics. London, Newnes, 1943. 240 pp. 8s. 6d.

DEMING, W. E. Statistical adjustment of data. New York, Wiley; London, Chapman and Hall, 1943. 10+261 pp. $3.50.

DIMICK, C. E., and HURD, C. C. Mathematics for mariners. New York, Van Nostrand, 1943. 7+254 pp. $2.75.

HURD, C. C. See DIMICK, C. E.

KELLS, L. M., KERN, W. F., BLAND, J. R., and ORLEANS, J. B. Elements of trigonometry. Plane and spherical with applications. New York and London, McGraw-Hill, 1943. 10+363 pp. $1.80.

KERN, W. F. See KELLS, L. M.

LANDOLT, M. Grösse, Masszahl und Einheit. Zurich, Rascher, 1943. Fr. 5.80.

LEVY, S. H. Introductory artillery mathematics and anti-aircraft mathematics. Berkeley and Los Angeles, University of California Press; London, Cambridge University Press, 1943. 8+100 pp. 15s.

ORLEANS, J. B. See KELLS, L. M.

Table of circular and hyperbolic tangents and cotangents for radian arguments. New York, National Bureau of Standards, 1943. 28+412 pp. $5.00.

Table of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments. New York, National Bureau of Standards, 1943. 44+406 pp. $5.00.

Table of reciprocals of the integers from 100,000 through 200,009. New York, National Bureau of Standards, 1943. 8+204 pp. $4.00.

VAN DER WAERDEN, G. L. Moderne Algebra. Reprint of 2nd revised edition. Parts 1 and 2 in one volume. New York, Stechert, 1943. 496 pp. $6.00.